INVARIANT DIFFERENTIAL DERIVATIONS FOR REFLECTION GROUPS IN POSITIVE CHARACTERISTIC

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Abstract. Much of the captivating numerology surrounding finite reflection groups stems from Solomon's celebrated 1963 theorem describing invariant differential forms. Invariant differential derivations also exhibit fascinating numerology over the complex numbers linked to rational Catalan combinatorics. We explore the analogous theory over arbitrary fields, in particular, when the characteristic of the underlying field divides the order of the acting reflection group and the conclusion of Solomon's Theorem may fail. Using results of Broer and Chuai, we give a Saito criterion (Jacobian criterion) for finding a basis of differential derivations invariant under a finite group that distinguishes certain cases over fields of characteristic 2. We show that the reflecting hyperplanes lie in a single orbit and demonstrate a duality of exponents and coexponents when the transvection root spaces of a reflection group are maximal. A set of basic derivations are used to construct a basis of invariant differential derivations with a twisted wedging in this case. We obtain explicit bases for the special linear groups $\mathrm{SL}(n, q)$ and general linear groups $\mathrm{GL}(n, q)$, and all groups in between.

1. Introduction

Solomon [22] showed that the set of differential forms invariant under the action of a complex reflection group forms a free exterior algebra. The situation is more subtle over an arbitrary field, especially when the characteristic of the underlying field $\mathbb{F}$ divides the order of the acting group, the so-called *modular setting*. Zalesskii and Serežkin [27] classified the irreducible reflection groups over fields of positive characteristic, but not every reflection group is the sum of irreducible reflection groups, and many interesting examples are reducible with nondiagonalizable reflections. Hartmann [11] showed that the conclusion of Solomon's Theorem holds for a group generated by diagonalizable reflections whose ring of invariant polynomials forms a polynomial algebra. Hartmann and the second author [13] extended this work to exhibit the space of invariant differential forms as a free exterior algebra via a twisted wedge product when the transvection root spaces are maximal. Such groups include $\mathrm{SL}_n(\mathbb{F}_q)$ and $\mathrm{GL}_n(\mathbb{F}_q)$ for a finite field $\mathbb{F}_q$, and we explore these groups as analogs of Coxeter and well-generated complex reflection groups. We assume all reflection groups are finite.

Recently, attention has turned to *differential derivations* as their invariants under a reflection group arise in Catalan combinatorics with connections to rational Cherednik algebras, symplectic reflection algebras, and Lie theory (e.g., see [9, 2, 3, 18, 1, 16, 8, 17]). The differential derivations invariant under the action of a well-generated complex reflection group constitute a free module over a certain subalgebra of the invariant differential forms, and associated Hilbert series give Kirkman numbers (see [16, 17]).

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We investigate the case over an arbitrary field $\mathbb{F}$ here. We examine the set $(S\otimes\wedge V^*\otimes V)^G$ of differential derivations invariant under a finite group G acting linearly on a finite-dimensional vector space $V=\mathbb{F}^n$, with symmetric algebra $S=S(V^*)$, a polynomial ring. We include the modular setting when $\mathrm{char}(\mathbb{F})$ divides $|G|$. Broer and Chuai [6] used ramifications over prime ideals to give a general Jacobian criterion. This criterion requires a full description of the invariant theory for groups fixing a single hyperplane. Finding this description may be trivial when all group elements are diagonalizable but often is a sticking point when working over arbitrary fields. Here, we require a rigorous analysis of the actions of transvections on differential derivations (see Appendix A). We develop a Saito criterion in terms of pointwise stabilizers for determining whether a set of homogeneous elements is a basis:

Theorem 1.1. *Consider a finite group $G\subset GL(V)$ acting on $V=\mathbb{F}^n$. For a set $\mathcal{B}$ of $n\binom{n}{k}$ homogeneous elements in $(S\otimes\wedge^k V^*\otimes V)^G$, the following are equivalent:*

a) $(S\otimes\wedge^k V^*\otimes V)^G$ is a free S^G-module with basis $\mathcal{B}$.

b) The coefficient matrix of $\mathcal{B}$ has determinant $Q^{\binom{n-1}{k}}Q_{\det}^{(n-1)\binom{n-1}{k-1}}Q_k$ up to a nonzero scalar.

c) $\mathcal{B}$ is independent over $\mathcal{F}(S)$ and $\displaystyle\sum_{\eta\in\mathcal{B}}\deg\eta=\sum_{H\in\mathcal{A}}\binom{n-1}{k}+(e_H-1)(n-1)\binom{n-1}{k-1}+e_H a_{H,k}\,.$

Here, $\mathcal{F}(S)$ is the field of fractions of S, e_H records the maximal order of a diagonalizable reflection in G about each H in the collection $\mathcal{A}$ of reflecting hyperplanes of G, the polynomial Q_k in S (see Eq. (3.4)) depends on the transvection root space of each H, and the nonnegative integers $a_{H,k}$ (see Eq. (3.5)) depend additionally on the characteristic of $\mathbb{F}$ in a subtle way.

We argue that reflection groups with transvection roots spaces all maximal, such as groups G with $\mathrm{SL}_n(\mathbb{F}_q)\subset G\subset\mathrm{GL}_n(\mathbb{F}_q)$ (see Section 9), serve as analogues of the duality (well-generated) complex reflection groups with *Coxeter number* given as the number of reflecting hyperplanes times the maximal order of a diagonalizable reflection in the group (see Remark 4.7 and Remark 5.7). The following result provides the structure of the invariant differential derivations for this class of reflection groups.

Theorem 1.2. *Let $G\subset GL(V)$ be a reflection group with transvection root spaces all maximal and* $\mathrm{char}\,\mathbb{F}\neq 2$*. Suppose $(S\otimes V)^G$ is a free S^G-module with basic derivations $\theta_1,\ldots,\theta_n$ and dual 1-forms $\omega_1,\ldots,\omega_n$. Then $(S\otimes\wedge V^*\otimes V)^G$ is a free S^G-module with basis*

$$\big\{d\theta_E\big\}\cup\big\{\omega_I^{\curlywedge}\,\theta_1,\ldots,\omega_I^{\curlywedge}\,\theta_n : I\subset[n]\big\}\setminus\big\{\omega_r\theta_r\big\}\quad \textit{for any } r=1,\ldots,n.$$

We use the exterior derivative of the Euler derivation, $d\theta_E=\sum_{i=1}^n 1\otimes x_i\otimes v_i$, dual 1-forms $\omega_1,\ldots,\omega_n$ constructed via an operator related to the Hodge dual (see Proposition 5.3), and twisted wedge products $\omega_I^{\curlywedge}$ (see Eq. (4.4)).

Example 1.3. For the reflection group $G=\langle\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\rangle$ acting on $V=\mathbb{F}^2$ with $\mathrm{char}\,\mathbb{F}=p>2$,

basic derivations $\theta_1=1\otimes v_1,\quad \theta_2=x_1\otimes v_1+x_2\otimes v_2$ generate $(S\otimes V)^G$ and

dual 1-forms $\omega_1=x_2\otimes x_1-x_1\otimes x_2,\quad \omega_2=-1\otimes x_2$ generate $(S\otimes V^*)^G$

as free S^G-modules. Then $(S\otimes\wedge V^*\otimes V)^G$ is a free S^G-module with basis

$$\theta_1=1\otimes 1\otimes v_1,\quad \theta_2=x_1\otimes 1\otimes v_1+x_2\otimes 1\otimes v_2,\quad d\theta_E=1\otimes x_1\otimes v_1+1\otimes x_2\otimes v_2,$$
$$\omega_1\theta_1=x_2\otimes x_1\otimes v_1-x_1\otimes x_2\otimes v_1,\quad \omega_2\theta_1=-1\otimes x_2\otimes v_1,$$
$$\omega_1\theta_2=x_1x_2\otimes x_1\otimes v_1+x_2^2\otimes x_1\otimes v_2-x_1^2\otimes x_2\otimes v_1-x_1x_2\otimes x_2\otimes v_2,$$
$$(\omega_1\curlywedge\omega_2)\theta_1=-1\otimes x_1\wedge x_2\otimes v_1,\quad (\omega_1\curlywedge\omega_2)\theta_2=-x_1\otimes x_1\wedge x_2\otimes v_1-x_2\otimes x_1\wedge x_2\otimes v_2.$$

Outline. In Section 2, we recall various properties of reflection groups and hyperplane arrangements and relate derivations and differential forms to differential derivations. We give Saito criteria for invariant derivations and 1-forms in Section 3. We then derive a Saito criterion for invariant differential derivations for all finite groups using an extensive analysis of the actions of transvections in Appendix A. In Sections 4–9, we focus on reflection groups whose transvection root spaces are maximal. We show the hyperplanes all lie in the same orbit, recall a twisted wedge product, and identify the semi-invariant differential forms in Section 4. In Section 5, we show how to construct a set of basic 1-forms when a set of basic derivations is known, and vice versa, demonstrating a duality of exponents and coexponents of the group. The structure of the set of invariant differential derivations when the characteristic of the base field is not 2 is given in Section 6 whereas Section 7 analyzes the characteristic 2 case. Section 8 considers groups acting on vector spaces over prime fields and Section 9 considers $\mathrm{SL}_n(\mathbb{F}_q)$, $\mathrm{GL}_n(\mathbb{F}_q)$, and all groups in between.

2. Background and Notation

We fix a finite-dimensional vector space $V = \mathbb{F}^n$ over a field $\mathbb{F}$ of arbitrary characteristic, $n \geq 1$. Let $S := S(V^*)$ be the symmetric algebra of V^* which we identify with the polynomial ring $\mathbb{F}[V] = \mathbb{F}[x_1, \ldots, x_n]$ for a basis $x_1, \ldots, x_n$ of V^*. We use $\mathcal{F}(S)$ for the fraction field of S. Let $G \subset \mathrm{GL}(V)$ be a finite group acting on V and consider the usual dual action on V^* (given by the inverse transpose of the matrix recording the action on V) which extends to an action on S by automorphisms. We write $a \doteq b$ to indicate a and b are equal up to a scalar in $\mathbb{F}^\times$. Note that all tensor products are taken over $\mathbb{F}$.

Invariants. For any $\mathbb{F}G$-module M, we write M^G for the invariants in M and

$$M^G_\chi = \{m \in M : g(m) = \chi(g)m \text{ for all } g \in G\} \qquad \text{for the } \chi\text{-invariants},$$

the space of semi-invariants with respect to a linear character $\chi : G \to \mathbb{F}^\times$ of G. We write $\det = \det_V : G \to \mathbb{F}^\times$ for the determinant character of G acting on V.

The space $S \otimes M$ is an S-module through multiplication in the first tensor component. Likewise, the space of invariants $(S \otimes M)^G$ is an S^G-module, necessarily of rank $\dim_{\mathbb{F}}(M)$ (e.g., see [4] or [6]), and we seek S^G-module bases when these invariant spaces are free.

Reflections. Recall that a *reflection* on a vector space $V = \mathbb{F}^n$ is a nonidentity invertible linear transformation that fixes pointwise a subspace of V of codimension 1, called the *reflecting hyperplane* of the transformation. A *reflection group* is a group generated by reflections, and we assume all reflection groups are finite. There are two types of reflections: diagonalizable reflections and *transvections* (nondiagonalizable). Note that $\mathrm{order}(s)$ and $\mathrm{char}\,\mathbb{F}$ are coprime and $\det(s)$ lies in $\mathbb{F}^\times$ when s is a diagonalizable reflection, whereas $\mathrm{order}(s) = \mathrm{char}\,\mathbb{F}$ and $\det(s) = 1$ when s is a transvection (see [21]).

Reflection arrangement of a finite group. We say a hyperplane H in V is a *reflecting hyperplane of* G when there is some reflection in G about H. We denote the (possibly empty) collection of all reflecting hyperplanes of G by $\mathcal{A} = \mathcal{A}(G)$ and note that $\mathcal{A}(G) = \mathcal{A}(W)$ for W the subgroup of G generated by the reflections in G.

Pointwise stabilizers of reflecting hyperplanes. We denote the pointwise stabilizer of each reflecting hyperplane H of G by $G_H = \{g \in G : g|_H = 1\}$. The transvections in G_H along with the identity form a normal subgroup K_H of G:

$$K_H = \ker\big(\det : G_H \to \mathbb{F}^\times\big)\,.$$

We set $e_H = |G_H : K_H|$ and observe that $G_H = \langle K_H, s_H\rangle \cong K_H \rtimes \mathbb{Z}/e_H\mathbb{Z}$, where s_H is a diagonalizable reflection in G about H of maximal order e_H when $e_H \neq 1$ and $s_H = 1_G$ when $e_H = 1$.

Root vectors. For each reflecting hyperplane H of G, we fix a linear form ℓ_H in V^* with $\ker \ell_H = H$. Each reflection s in G about H is then defined by its *root vector* v_s spanning $\mathrm{Im}(s-1) \subset V$ with respect to ℓ_H, see [21]:

$$s(v) = v + \ell_H(v)v_s \quad \text{for all } v \text{ in } V.$$

Note that a reflection s about H is a transvection exactly when its root vector v_s lies *in* H.

Root spaces. The *root space* $\mathcal{R}_H$ of a reflecting hyperplane H of G is the $\mathbb{F}$-vector space spanned by all of the root vectors of the reflections in G about H. The *transvection root space* of H (see [13]) is the space $\mathcal{R}_H \cap H$ spanned by the root vectors of the transvections in G about H. We denote its dimension by b_H:

$$b_H := \dim_{\mathbb{F}}(\mathcal{R}_H \cap H) = \dim_{\mathbb{F}} \mathbb{F}\text{-span}\{v_s : s \text{ is a transvection in } G \text{ about } H\}\,.$$

If the transvection root space of H is all of H, i.e., $\mathcal{R}_H\ \cap H = H$, then $b_H = n-1$ and we say the transvection root space is *maximal*. Often all of the transvection root spaces for G are maximal, as is the case, for example, when $\mathrm{SL}_n(\mathbb{F}_q) \subset G \subset \mathrm{GL}_n(\mathbb{F}_q)$.

Arrangement polynomials. We consider the arrangement-defining polynomial Q in S and polynomials $Q_{\det}$ and $Q(\tilde{\mathcal{A}})$ (see [13]) which vanish on some reflecting hyperplanes or are 1:

$$Q := \prod_{H\in\mathcal{A}} \ell_H\,, \quad Q_{\det} := \prod_{H\in\mathcal{A}} \ell_H^{e_H-1}\,, \quad \text{and} \quad Q(\tilde{\mathcal{A}}) := \prod_{H\in\mathcal{A}} \ell_H^{e_H b_H}\,.$$

These polynomials depend only upon G up to a scalar in $\mathbb{F}^\times$. Recall that $Q_{\det}$ divides any polynomial that is semi-invariant with respect to the linear character $\det = \det_V : W \to \mathbb{F}^\times$ of the subgroup W generated by the reflections in G. In fact (see Eq. (4.8), [23], [14], [20]), for $Q_{\det^{-1}} = \prod_{H\in\mathcal{A}: e_H\neq 1} \ell_H$,

$$S^W_{\det} = Q_{\det}\, S^W \quad \text{and} \quad S^W_{\det^{-1}} = Q_{\det^{-1}}\, S^W\,. \tag{2.1}$$

Vector space basis for one hyperplane. For any reflecting hyperplane H of G, we may choose a convenient basis $v_1, \ldots, v_n$ of V with dual basis $x_1, \ldots, x_n$ of V^* so that $v_1, \ldots, v_{n-1}$ span H and $\ell_H = x_n$. In fact, we may choose $v_1, \ldots, v_{b_H}$ to be root vectors of transvections $t_1, \ldots, t_{b_H}$ in G about H and $v_n \notin H$ to be an eigenvector of s_H with eigenvalue $\lambda \in \mathbb{F}^\times$ of order e_H. With respect to this basis,

$$t_m = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & 1 \\ & & & \ddots \\ & & & & 1 \end{pmatrix} \leftarrow m^{\text{th}} \text{ row} \quad \text{for } 1 \le m \le b_H \quad \text{and} \quad s_H = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \lambda \end{pmatrix}. \tag{2.2}$$

Note that $e_H = 1$, $s_H = 1_G$, and $\lambda = 1$ when G contains no diagonalizable reflections about H. When $\mathbb{F} = \mathbb{F}_p$, G_H is precisely $\langle t_1, \ldots, t_{b_H}, s_H\rangle$, so $|G_H| = e_H\, p^{b_H}$. In general, however, G_H may contain more transvections about H (see Lemma 2.1 in [12]).

Example 2.3. Let $V = \mathbb{F}_5^2$ with standard basis v_1, v_2 and dual basis x_1, x_2 of V^*. Consider the group $G = \langle t, s, g\rangle$ generated by

$$t = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \quad s = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \quad \text{and} \quad g = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 2 \end{smallmatrix}\right).$$

The subgroup of G generated by its reflections is $W = \langle t, s\rangle$, which fixes a single hyperplane $H = \ker x_2$. The transvection root space of H has dimension $b_H = 1$ (i.e., $\mathcal{R}_H \cap H = H$) and the maximal order of a diagonalizable reflection is $e_H = 2$. So $Q = Q_{\det} = x_2$ and $Q(\tilde{\mathcal{A}}) = x_2^2$.

Derivations and differential forms. We identify the set of S-derivations Der_S on V with $S \otimes V$, identify the set of differential forms on V with $S \otimes \wedge V^*$, and consider the S-module $S \otimes \wedge V^* \otimes V$ of *differential derivations* on V (otherwise called *mixed forms*, see [16]):

$$\begin{aligned} &S \otimes V && (\textit{derivations}),\\ &S \otimes \wedge V^* && (\textit{differential forms}),\\ &S \otimes \wedge V^* \otimes V && (\textit{differential derivations}). \end{aligned}$$

Consider a basis $v_1, \ldots, v_n$ of V with dual basis $x_1, \ldots, x_n$ of V^* and a set $\omega_1, \ldots, \omega_n \in S \otimes V^*$. For any subset $I = \{i_1, \ldots, i_k\}$ of $[n] = \{1, \ldots, n\}$ with $i_1 < \ldots < i_k$, we set

$$\begin{aligned} v_I &:= v_{i_1} \wedge \cdots \wedge v_{i_k} \in \wedge^k V\,,\\ x_I &:= x_{i_1} \wedge \cdots \wedge x_{i_k} \in \wedge^k V^*\,, \qquad (2.4)\\ \omega_I &:= \omega_{i_1} \wedge \cdots \wedge \omega_{i_k} \in S \otimes \wedge^k V^*\,, \end{aligned}$$

with $v_I = 1$, $x_I = 1$, and $\omega_I = 1 \otimes 1$ for the empty set $I = \varnothing$. To indicate subsets of size k, we write $I \in \binom{[n]}{k}$ for $I \subset [n]$ with $|I| = k$. We denote the volume form on V by $\mathrm{vol}_V = v_1 \wedge \cdots \wedge v_n \in \wedge^n V$ and the volume form on V^* by $\mathrm{vol}_{V^*} = x_1 \wedge \cdots \wedge x_n \in \wedge^n V^*$.

Differential derivations as a module over the differential forms. We view the set of differential derivations $S \otimes \wedge V^* \otimes V$ as a module over the set of differential forms $S \otimes \wedge V^*$ via multiplication in the first two tensor components: for f, f' in S, x_I, x_I' in $\wedge V^*$, and $v \in V$,

$$(f \otimes x_I)(f' \otimes x_I' \otimes v) := ff' \otimes x_I \wedge x_I' \otimes v. \qquad (2.5)$$

Embedding derivations into the differential derivations. We embed the set of derivations into the set of differential derivations:

$$S \otimes V \hookrightarrow S \otimes \wedge V^* \otimes V, \qquad f \otimes v \mapsto f \otimes 1 \otimes v.$$

This embedding together with the module structure of Eq. (2.5) allows us to multiply a differential form and a derivation to construct a differential derivation, with the G-action preserved:

$$\begin{aligned} (S \otimes \wedge V^*) \times (S \otimes V) &\longrightarrow S \otimes \wedge V^* \otimes V,\\ (f \otimes x_I) \;\times (f' \otimes v) &\longmapsto ff' \otimes x_I \otimes v\,. \end{aligned}$$

Degree and rank. We assign $\deg x_i = 1$ for all i so that $S = \bigoplus_i S_i$ is graded by the usual polynomial degree and G acts by graded automorphisms. For any $\mathbb{F}G$-module M, we say the elements of $S_i \otimes M$ are *homogeneous of polynomial degree i*. We say elements in $S \otimes \wedge^k V^*$ and in $S \otimes \wedge^k V^* \otimes V$ have *rank k*. Thus 1-forms are differential forms of rank 1, and for homogeneous f in S, $I \subset [n]$, and $v \in V$,

$$\deg(f \otimes x_I \otimes v) = \deg(f) \quad \text{and} \quad \mathrm{rank}(f \otimes x_I \otimes v) = \mathrm{rank}(x_I) = |I|\,.$$

For any $\mathbb{F}G$-module M, one may choose a homogeneous basis of the S^G-module $(S \otimes M)^G$ when free by the graded Nakayama Lemma (e.g., see [7, Section 2.10] or [21, Corollary 5.2.5]), and the set of polynomial degrees of elements in such a basis is independent of this choice.

Euler derivation. Recall that the *Euler derivation* $\theta_E := \sum_{i=1}^n x_i \otimes v_i$ is invariant under any linear group action. We use the invariant differential derivation $d\theta_E$ (see [16]):

$$d\theta_E = 1 \otimes x_1 \otimes v_1 + \cdots + 1 \otimes x_n \otimes v_n \in (S \otimes V^* \otimes V)^G .$$

Coefficient matrix. For any $\mathbb{F}G$-module M, we define the coefficient matrix of $\omega_1, \ldots, \omega_\ell$ in $S \otimes M$ with respect to an ordered basis $z_1, \ldots, z_m$ of M as usual by

$$\operatorname{Coef}(\omega_1, \ldots, \omega_\ell) := \{f_{ij}\}_{\substack{1 \le i \le \ell \\ 1 \le j \le m}} \in M_{\ell \times m}(S), \qquad \text{where } \omega_i = \sum_{j=1}^m f_{ij} \otimes z_j \text{ for } 1 \le i \le \ell .$$

For any unordered set $\mathcal{B} \subset S \otimes M$ with $|\mathcal{B}| = m$, the determinant $\det \operatorname{Coef}(\mathcal{B})$ is defined up to a sign and is nonzero precisely when $\mathcal{B}$ is independent over $\mathcal{F}(S)$. Notice that the coefficient vector of a differential derivation arising as the product of a differential form and a derivation is just the tensor product of the respective coefficient vectors: for any $\omega = \sum_I f_I \otimes x_I$ in $S \otimes \wedge^k V^*$ and $\theta = \sum_{j=1}^n f'_j \otimes v_j$ in $S \otimes V$,

$$\omega\theta = \sum_{I,j} f_I f'_j \otimes x_I \otimes v_j \quad \text{ with } \operatorname{Coef}(\omega\theta) = \operatorname{Coef}(\omega) \otimes \operatorname{Coef}(\theta) = \big(f_{I_1}, \ldots, f_{I_m}\big) \otimes \big(f'_1, \ldots, f'_n\big)$$

with respect to a fixed ordered basis $x_I \otimes v_j$ of $\wedge^k V^* \otimes V$ arising from ordered bases $v_1, \ldots, v_n$ of V and $x_{I_1}, \ldots, x_{I_m}$ of $\wedge^k V^*$ with $m = \binom{n}{k}$. This extends to subsets of differential forms $\mathcal{B} \subset S \otimes \wedge V^*$ and derivations $\mathcal{B}' \subset S \otimes V$: with the appropriate orderings,

$$\operatorname{Coef}(\omega\theta : \omega \in \mathcal{B}, \theta \in \mathcal{B}') = \operatorname{Coef}(\mathcal{B}) \otimes \operatorname{Coef}(\mathcal{B}') .$$

This fact immediately implies the following observation since $\{\omega_I : I \subset [n]\}$ is independent over $\mathcal{F}(S)$ whenever $\{\omega_1, \ldots, \omega_n\} \subset S \otimes V^*$ is independent.

Lemma 2.6. *If $\{\theta_1, \ldots, \theta_n\} \subset S \otimes V$ and $\{\omega_1, \ldots, \omega_n\} \subset S \otimes V^*$ are both $\mathcal{F}(S)$-independent, then so is*

$$\big\{\omega_I\, \theta_j : I \in \tbinom{[n]}{k}, 1 \le j \le n\big\} \quad \subset S \otimes \wedge^k V^* \otimes V .$$

3. Saito/Jacobian Criterion

We consider a finite group $G \subset \mathrm{GL}(V)$ acting on $V = \mathbb{F}^n$. We give criteria for finding S^G-bases of invariant derivations $(S \otimes V)^G$ and invariant 1-forms $(S \otimes V^*)^G$ before examining invariant differential derivations $(S \otimes \wedge V^* \otimes V)^G$.

Solomon's Theorem. Solomon [22] showed that when G is a reflection group acting on $V = \mathbb{C}^n$, the set of invariant differential forms $(S \otimes \wedge V^*)^G$ is a free exterior algebra over S^G generated by $df_1, \ldots, df_n$ for any polynomials $f_1, \ldots, f_n$ with $S^G = \mathbb{C}[f_1, \ldots, f_n]$:

$$(S \otimes \wedge V^*)^G = \bigwedge\nolimits_{S^G} \{\omega_1, \ldots, \omega_n\} \quad \text{with } \omega_i = df_i \text{ for } 1 \le i \le n .$$

For a reflection group G acting on $V = \mathbb{F}^n$ with $\operatorname{char} \mathbb{F}$ dividing $|G|$, the ring of invariant polynomials S^G may not be a polynomial algebra. Even when $S^G = \mathbb{F}[f_1, \ldots, f_n]$ for some f_i in S^G (i.e., the action is *coregular*, see [5]), the exterior derivatives df_i do not generate $(S \otimes \wedge V^*)^G$ as an exterior algebra when G contains transvections: Hartmann [11] showed that the conclusion of Solomon's Theorem holds if and only if S^G is a polynomial algebra and G contains no transvections.

Saito criteria for invariant derivations and 1-forms. Criteria for finding bases of invariant derivations and invariant 1-forms under the action of a finite linear group G relies on the pointwise stabilizer subgroups G_H of each of the reflecting hyperplanes $H \in \mathcal{A}$ of G. Thus we begin with groups fixing a single hyperplane; a more general criteria will follow from [6, Theorem 3].

We use the 1-forms from [13, Remark 13] and provide a short direct proof for derivations. For each reflecting hyperplane H of G, recall that e_H is the maximal order of a diagonalizable reflection in G about H (or $e_H = 1$ if none exist) and b_H is the dimension of the transvection root space of H. We consider the exterior product of derivations $\theta_1 \wedge \cdots \wedge \theta_n$ in $S \otimes \wedge^n V$ and of 1-forms $\omega_1 \wedge \cdots \wedge \omega_n$ in $S \otimes \wedge^n V^*$.

Lemma 3.1. *Suppose a nontrivial finite group $G \subset GL(V)$ fixes pointwise a hyperplane $H = \ker \ell_H$ in $V = \mathbb{F}^n$. Then $(S \otimes V)^G$ and $(S \otimes V^*)^G$ are free S^G-modules, and for any $\theta_1, \ldots, \theta_n$ in $(S \otimes V)^G$ and any $\omega_1, \ldots, \omega_n$ in $(S \otimes V^*)^G$,*

- *$\theta_1, \ldots, \theta_n$ are an S^G-basis of $(S \otimes V)^G$ if and only if $\theta_1 \wedge \cdots \wedge \theta_n \doteq \ell_H \operatorname{vol}_V$, and*
- *$\omega_1, \ldots, \omega_n$ are an S^G-basis of $(S \otimes V^*)^G$ if and only if $\omega_1 \wedge \cdots \wedge \omega_n \doteq \ell_H^{e_H b_H + e_H - 1} \operatorname{vol}_{V^*}$.*

Proof. We exhibit an explicit S^G-basis $\theta_1, \ldots, \theta_n$ of $(S \otimes V)^G$ with $\theta_1 \wedge \cdots \wedge \theta_n \doteq \ell_H \operatorname{vol}_V$ and an explicit S^G-basis $\omega_1, \ldots, \omega_n$ of $(S \otimes V^*)^G$ with $\omega_1 \wedge \cdots \wedge \omega_n \doteq \ell_H^{e_H b_H + e_H - 1} \operatorname{vol}_{V^*}$. The result then follows from [6, Theorem 3] (see also [6, Proposition 6]). We use the basis $v_1, \ldots, v_n$ of V with dual basis $x_1, \ldots, x_n$ of V^* of Eq. (2.2), so $\ell_H = x_n$, and consider the invariants

$$\theta_i = \begin{cases} 1 \otimes v_i & \text{for } 1 \le i < n, \\ \sum_{j=1}^n x_j \otimes v_j & \text{for } i = n, \end{cases} \quad \text{and} \quad \omega_i = \begin{cases} x_n^{e_H} \otimes x_i - x_i x_n^{e_H - 1} \otimes x_n & \text{for } 1 \le i \le b_H, \\ 1 \otimes x_i & \text{for } b_H < i < n, \\ x_n^{e_H - 1} \otimes x_n & \text{for } i = n\,. \end{cases}$$

Then $\omega_1 \wedge \cdots \wedge \omega_n = x_n^{e_H b_H + e_H - 1} \operatorname{vol}_{V^*}$, and the ω_i are an S^G-basis of $(S \otimes V^*)^G$ by [13].

Fix some $\theta = \sum_i h_i \otimes v_i$ in $(S \otimes V)^G$. For $g \neq 1_G$ in G with root vector $(\alpha_1, \ldots, \alpha_n)$, so $g(v_n) = \sum_{i \neq n} \alpha_i v_i + (1 + \alpha_n) v_n$ with $\alpha_n \neq -1$, we equate polynomial coefficients of θ with those of $g(\theta)$ and conclude that

$$g(h_i) = \begin{cases} h_i - \frac{\alpha_i}{1+\alpha_n} h_n & \text{for } i \neq n\,, \\ \frac{1}{1+\alpha_n} h_n & \text{for } i = n\,, \end{cases} \qquad \text{while} \qquad g(x_i) = \begin{cases} x_i - \frac{\alpha_i}{1+\alpha_n} x_n & \text{for } i \neq n\,, \\ \frac{1}{1+\alpha_n} x_n & \text{for } i = n\,. \end{cases}$$

Note that $\alpha_j \neq 0$ for some j (as $g \neq 1_G$), so $h_n \doteq g(h_j) - h_j$ is divisible by x_n (see Lemma A.1). Also, g fixes $\frac{h_n}{x_n}$ and $h_i - \frac{h_n}{x_n} x_i$ for $i \neq n$. As g was arbitrary, these polynomials lie in S^G, and

$$\theta = \tfrac{h_n}{x_n} \theta_n + \sum_{i \neq n} (h_i - \tfrac{h_n}{x_n} x_i) \theta_i$$

lies in the S^G-span of the θ_i. As $\theta_1 \wedge \cdots \wedge \theta_n = x_n \operatorname{vol}_V \neq 0$, the θ_i are independent over $\mathcal{F}(S)$, and thus over S^G, and are an S^G-basis of $(S \otimes V)^G$. $\square$

Lemma 3.1 together with [6, Theorem 3] implies the following analog of the classical Saito Criterion (see [15, Corollary 6.61, Proposition 6.47]) for all finite linear groups, including those with transvections.

Theorem 3.2. *Consider a finite group $G \subset GL(V)$ acting on $V = \mathbb{F}^n$. For homogeneous $\theta_1, \ldots, \theta_n$ in $(S \otimes V)^G$, the following are equivalent:*

a) $(S \otimes V)^G$ is a free S^G-module with basis $\theta_1, \ldots, \theta_n$.
b) $\theta_1 \wedge \cdots \wedge \theta_n \doteq Q \operatorname{vol}_V$.
c) $\theta_1, \ldots, \theta_n$ are independent over $\mathcal{F}(S)$ and $\sum_{i=1}^n \deg \theta_i = \deg Q = |\mathcal{A}|$.

For homogeneous $\omega_1, \ldots, \omega_n$ in $(S \otimes V^*)^G$, the following are equivalent:

a) $(S \otimes V^*)^G$ is a free S^G-module with basis $\omega_1, \ldots, \omega_n$.
b) $\omega_1 \wedge \cdots \wedge \omega_n \doteq Q(\tilde{\mathcal{A}}) Q_{\det} \operatorname{vol}_{V^*}$.
c) $\omega_1, \ldots, \omega_n$ are independent over $\mathcal{F}(S)$ and $\sum_{i=1}^n \deg \omega_i = \sum_{H \in \mathcal{A}} \big(e_H b_H + e_H - 1\big)$.

We call $\theta_1, \ldots, \theta_n$ satisfying the equivalent conditions of the last theorem *basic derivations* and call $\omega_1, \ldots, \omega_n$ satisfying the equivalent conditions of the last theorem *basic* 1*-forms*.

Saito criterion for invariant differential derivations. Now we turn our attention to establishing a Saito criterion for $(S \otimes \wedge^k V^* \otimes V)^G$. This requires a detailed analysis of the action of transvections relegated to Appendix A. Such care is not required over fields of characteristic zero as all reflections are diagonalizable. Within the analysis, we distinguish those hyperplanes of G whose pointwise stabilizers G_H consist of exactly one transvection and the identity. Define δ_H to be 1 in this case and 0 otherwise:

$$\delta_H := \begin{cases} 1 & \text{if } G_H = \{1_G, \text{one transvection}\}\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{3.3}$$

Note that when $\operatorname{char} \mathbb{F} \neq 2$, any transvection and its inverse are distinct, so

$$\delta_H = 0 \text{ for all } H \in \mathcal{A} \qquad \text{whenever } \operatorname{char} \mathbb{F} \neq 2\,.$$

Additionally, for each $0 \le k \le n$ corresponding to the rank of a differential derivation, we define a polynomial which depends only upon G up to a scalar in $\mathbb{F}^\times$,

$$Q_k := \prod_{H \in \mathcal{A}} \ell_H^{\,e_H\, a_{H,k}}\,, \tag{3.4}$$

in terms of integers $a_{H,k} \ge 0$ depending on the pointwise stabilizer G_H of each H in $\mathcal{A}$:

$$a_{H,k} := (n - \delta_H)\left(\binom{n-1}{k} - \binom{n-b_H-1}{k}\right) + \binom{n-1}{k-1} - \binom{n-b_H-1}{k-1}\,. \tag{3.5}$$

Here, $\binom{a}{b} = 0$ if $a < b$ or $b < 0$.

Remark 3.6. In the nonmodular setting, the group G contains no transvections and $b_H = 0$ for every reflecting hyperplane (minimal transvection root spaces), so each $a_{H,k} = 0$ and

$$Q_k = 1 \quad \text{ when } \operatorname{char} \mathbb{F} \text{ and } |G| \text{ are coprime}\,.$$

On the other end of the spectrum, we will see in Section 4 that if $b_H = n-1$ for every reflecting hyperplane of G (maximal transvection root spaces), then the reflecting hyperplanes are in a single G-orbit and there are fixed nonnegative integers e, b, δ, and a_k with $e = e_H$, $b = b_H$, $\delta = \delta_H$, and $a_k = a_{H,k}$ for every reflecting hyperplane H, and

$$Q_k = (Q Q_{\det})^{a_k} \quad \text{ with } \quad a_k = \begin{cases} 0 & \text{when } k = 0\,, \\ (n-\delta)(n-1) & \text{when } k = 1\,, \\ (n-\delta)\binom{n-1}{k} + \binom{n-1}{k-1} & \text{when } k \ge 2\,. \end{cases}$$

Now we establish a polynomial that divides the determinant of the coefficient matrix of any potential basis of invariant differential derivations of fixed rank. Compare with [16, Lemma 6.1] in the nonmodular case, where $Q_k = 1$ for all $k \geq 0$. The analysis required here (relegated to the appendix) is more nuanced because of the existence of transvections.

Lemma 3.7. *Consider a finite group $G \subset GL(V)$. For any set $\mathcal{B}$ of $n\binom{n}{k}$ elements in $(S \otimes \wedge^k V^* \otimes V)^G$, the determinant of* $\operatorname{Coef}(\mathcal{B})$ *is divisible by the polynomial*

$$Q^{\binom{n-1}{k}} Q_{\det}^{(n-1)\binom{n-1}{k-1}} Q_k \,.$$

Proof. Fix a reflecting hyperplane $H = \ker \ell_H$ of G. By Lemma A.8 in Appendix A, $\det \operatorname{Coef}(\mathcal{B})$ is divisible by ℓ_H to the power $\binom{n-1}{k} + (e_H - 1)(n-1)\binom{n-1}{k-1} + e_H a_{H,k}$. As the linear forms ℓ_H are pairwise coprime for $H \in \mathcal{A}$, the claim follows. □

We establish a Saito criterion for invariant differential derivations in Theorem 3.9, and the proof depends on an analysis for pointwise stabilizers G_H of reflecting hyperplanes $H \in \mathcal{A}$. As for the derivations and differential forms, we first require a criterion for the case of a group fixing one hyperplane. Recall that we write $I \in \binom{[n]}{k}$ when $I \subset [n] = \{1, \dots, n\}$ with $|I| = k$.

Proposition 3.8. *Suppose a nontrivial finite group $G \subset GL(V)$ fixes pointwise a hyperplane $H = \ker \ell_H$ in $V = \mathbb{F}^n$. Then $(S \otimes \wedge^k V^* \otimes V)^G$ is a free S^G-module for all k, and elements $\eta_1 \dots, \eta_m$ are a basis if and only if $m = n\binom{n}{k}$ and*

$$\det \operatorname{Coef}(\eta_1, \dots, \eta_m) \doteq \ell_H^{\binom{n-1}{k} + (e_H - 1)(n-1)\binom{n-1}{k-1} + e_H a_{H,k}} \,.$$

Proof. We abbreviate $\ell = \ell_H$, $e = e_H$, $b = b_H$, $\delta = \delta_H$, $a_k = a_{H,k}$, and use the basis $v_1, \dots, v_n$ of V with dual basis $x_1, \dots, x_n$ of V^* of Eq. (2.2) so that $\ell = x_n$ and x_n^e is G-invariant. We also use the basic derivations θ_i and basic 1-forms ω_i from the proof of Lemma 3.1. For a fixed k, consider the subset of invariant differential derivations

$$\mathcal{B}_k = \left\{ \tilde{\omega}_I \, \theta_j : I \in \tbinom{[n]}{k},\ 1 \leq j \leq n \text{ with } n \notin I \text{ or } j \neq n \right\} \cup \left\{ \tilde{\omega}_I \, d\theta_E : I \in \tbinom{[n]}{k-1} \text{ with } n \notin I \right\},$$

where $\tilde{\omega}_I = \omega_I / x_n^{e\, m(I)}$ for $m(I) = \max\left\{0\,,\, |I \cap \{1, \dots, b, n\}| - 1\right\}$. By Lemma 2.6, the set $\{\tilde{\omega}_I \, \theta_j : I \in \binom{[n]}{k}, 1 \leq j \leq n\}$ is independent over $\mathcal{F}(S)$. We argue that, for each I with $n \notin I$, we may replace $\tilde{\omega}_{I \cup \{n\}} \theta_n$ in this set by $\tilde{\omega}_I \, d\theta_E$ while maintaining $\mathcal{F}(S)$-independence to show that the resulting set $\mathcal{B}_k$ is also $\mathcal{F}(S)$-independent. Note that

$$x_n^e \, d\theta_E = \sum_{i=1}^{b} \omega_i \theta_i + \sum_{i=b+1}^{n-1} (x_n^e \omega_i \theta_i - x_i \omega_n \theta_i) + \omega_n \theta_n \,,$$

and thus, for $I \subset \{1, \dots, n-1\}$,

$$\tilde{\omega}_I \, d\theta_E = \frac{1}{x_n^e} \sum_{i=1}^{b} (\tilde{\omega}_I \wedge \omega_i) \, \theta_i + \frac{1}{x_n^e} \sum_{i=b+1}^{n-1} \left(x_n^e \, \tilde{\omega}_I \wedge \omega_i - x_i \, \tilde{\omega}_I \wedge \omega_n \right) \theta_i + \frac{1}{x_n^e} (\tilde{\omega}_I \wedge \omega_n) \, \theta_n \,.$$

Thus each $\tilde{\omega}_I \, d\theta_E$ lies in the $\mathcal{F}(S)$-span of $\{\tilde{\omega}_I \, \theta_j : I \in \binom{[n]}{k}, 1 \leq j \leq n\}$ with the coefficient of $\tilde{\omega}_{I \cup \{n\}} \theta_n$ nonzero when $n \notin I$. As the various sets $I \cup \{n\}$ with $n \notin I$ are distinct, $\mathcal{B}_k$ is $\mathcal{F}(S)$-independent.

First suppose $\delta = 0$ and set $\Delta_k = \binom{n-1}{k} + (e-1)(n-1)\binom{n-1}{k-1} + ea_k$. The module $(S \otimes \wedge^k V^* \otimes V)^G$ is free over S^G by [6, Proposition 6], say with basis $\mathcal{C}_k$. Each element of $\mathcal{B}_k$ lies in the S^G-span of $\mathcal{C}_k$ with polynomial coefficients recorded by some matrix M, and

$$\det \mathrm{Coef}(\mathcal{C}_k) \cdot \det(M) \ = \det \mathrm{Coef}(\mathcal{B}_k) \neq 0\,.$$

By Lemma 3.7, ℓ^{Δ_k} divides $\det \mathrm{Coef}(\mathcal{C}_k)$ in S (as $\delta = 0$), while a calculation confirms that $\deg(\det \mathrm{Coef}(\mathcal{B}_k)) = \Delta_k$. Hence $\det \mathrm{Coef}(\mathcal{C}_k) \doteq \ell^{\Delta_k}$ and [6, Theorem 3] implies the result.

Now suppose $\delta = 1$ and set $\Delta'_k = \binom{n-2}{k} + n\binom{n-2}{k-1} + \binom{n-2}{k-2}$. Here, $G = \{1_G, t_1\}$, $\mathrm{char}\,\mathbb{F} = 2$, and $e = b = 1$. Consider an alternate subset of invariant differential derivations

$$\begin{aligned}\mathcal{B}'_k =& \big\{\tilde{\omega}_I\, \theta_j : I \in \tbinom{[n]}{k},\ 1 \leq j \leq n \text{ with } I \cap \{1, n\} = \varnothing \text{ or } j \neq n\big\}\\ &\cup \big\{\tilde{\omega}_I\, d\theta_E : I \in \tbinom{[n]}{k-1} \text{ with } n \notin I\big\} \cup \big\{\tilde{\omega}_I\, \eta_0 : I \in \tbinom{[n]}{k-1} \text{ with } I \cap \{1, n\} = \varnothing\big\}\,,\end{aligned}$$

where $\eta_0 = x_1 \otimes x_1 \otimes v_1 + x_n \otimes x_1 \otimes v_n + x_1 \otimes x_n \otimes v_1 + x_1 \otimes x_n \otimes v_n$, which is G-invariant. We argue that, for each I with $I \cap \{1, n\} = \varnothing$, we may replace $\tilde{\omega}_{I \cup \{1\}} \theta_n$ in $\mathcal{B}_k$ by $\tilde{\omega}_I\, \eta_0$ while maintaining $\mathcal{F}(S)$-independence to show that $\mathcal{B}'_k$ is also $\mathcal{F}(S)$-independent. As $\mathrm{char}\,\mathbb{F} = 2$,

$$x_n\, \eta_0 = \sum_{i=2}^{n-1} x_i \omega_1 \theta_i + (x_1^2 + x_1 x_n)\omega_n \theta_1 + \omega_1 \theta_n\,,$$

and hence, for $I \subset \{2, \ldots, n-1\}$,

$$\tilde{\omega}_I\, \eta_0 = \sum_{i=2}^{n-1} \Big(\frac{\tilde{\omega}_I \wedge \omega_1}{x_n}\Big)\theta_i + (x_1^2 + x_1 x_n)\Big(\frac{\tilde{\omega}_I \wedge \omega_n}{x_n}\Big)\theta_1 + \Big(\frac{\tilde{\omega}_I \wedge \omega_1}{x_n}\Big)\theta_n\,.$$

Thus each $\tilde{\omega}_I\, \eta_0$ with $I \cap \{1, n\} = \varnothing$ lies in the $\mathcal{F}(S)$-span of a subset of $\mathcal{B}_k$ with nonzero coefficient of $\tilde{\omega}_{I \cup \{1\}} \theta_n$. As these subsets are disjoint for the various I with $I \cap \{1, n\} = \varnothing$, the set $\mathcal{B}'_k$ is $\mathcal{F}(S)$-independent and $\det \mathrm{Coef}(\mathcal{B}'_k) \neq 0$. A computation shows that Δ'_k is simultaneously the degree of $\det \mathrm{Coef}(\mathcal{B}'_k)$ and the degree of the polynomial in Lemma 3.7 (as $\delta = 1$). The claim then follows as in the previous case using [6, Theorem 3]. $\square$

Proposition 3.8 with [6, Theorem 3] then implies Theorem 1.1 of the introduction:

Theorem 3.9. *Consider a finite group $G \subset GL(V)$ acting on $V = \mathbb{F}^n$. For a set $\mathcal{B}$ of $n\binom{n}{k}$ homogeneous elements in $(S \otimes \wedge^k V^* \otimes V)^G$, the following are equivalent:*

a) $(S \otimes \wedge^k V^* \otimes V)^G$ is a free S^G-module with basis $\mathcal{B}$.

b) $\det \mathrm{Coef}(\mathcal{B}) \doteq Q^{\binom{n-1}{k}} Q_{\det}^{(n-1)\binom{n-1}{k-1}} Q_k$.

c) $\mathcal{B}$ is independent over $\mathcal{F}(S)$ and $\displaystyle\sum_{\eta \in \mathcal{B}} \deg \eta = \sum_{H \in \mathcal{A}} \tbinom{n-1}{k} + (e_H - 1)(n-1)\tbinom{n-1}{k-1} + e_H a_{H,k}\,.$

Example 3.10. Let $G \subset \mathrm{GL}(V)$ be a nontrivial finite group with $\dim V = 1$. Then G is cyclic say with generator s of order $e > 1$. Notice that s is a reflection fixing the hyperplane $H = \{0_V\}$, so G is a reflection group. All elements in G are diagonalizable, so G does not contain any transvections, and the transvection root space of H has dimension $0 = n - 1$ (so is maximal). Let v be a basis of V with dual basis x of V^*. Then $\omega = x^{e-1} \otimes x$ is an S^G-basis of $(S \otimes V^*)^G$ and $\theta = x \otimes v$ is an S^G-basis of $(S \otimes V)^G$ (see Theorem 3.2). Finally, θ and $d\theta_E$ are an S^G-basis of $(S \otimes \wedge V^* \otimes V)^G$ by Theorem 3.9. Note that ω and θ are dual in some sense, see Remark 5.5.

Remark 3.11. Note that one direction of Theorem 3.9 follows directly from a version of Solomon's 1963 original argument [22]. Indeed, (b) and (c) are equivalent by Lemma 3.7, and any $\mathcal{F}(S)$-independent ordered subset $\mathcal{B} = \{\eta_1, \dots, \eta_m\}$ with $m = n\binom{n}{k}$ spans $\mathcal{F}(S) \otimes M$ over $\mathcal{F}(S)$ for $M = \wedge^k V^* \otimes V$. Hence, after relabeling the basis elements $x_I \otimes v_j$ of M as $z_1, \dots, z_m$, we may write a fixed $\eta \in (S \otimes M)^G$ as

$$\eta = \sum_j f_j \otimes z_j \quad \text{ and as } \quad \eta = \sum_i h_i\, \eta_i \text{ for some } h_i \in \mathcal{F}(S).$$

Here, $\eta_i = \sum_j \operatorname{Coef}(\mathcal{B})_{ij} \otimes z_j$ (with each $\operatorname{Coef}(\mathcal{B})_{ij}$ in S) and $\mathbf{h} \cdot \operatorname{Coef}(\mathcal{B}) = \mathbf{f}$ for row vectors $\mathbf{h} = (h_1, \dots, h_m)$ and $\mathbf{f} = (f_1, \dots, f_m)$. Cramer's Rule implies that $h_i = \frac{\det \operatorname{Coef}(\mathcal{B})_{(i)}}{\det \operatorname{Coef}(\mathcal{B})}$, where $\operatorname{Coef}(\mathcal{B})_{(i)}$ is obtained by replacing the i-th row of $\operatorname{Coef}(\mathcal{B})$ by $\mathbf{f}$. Since $\eta_1 \wedge \cdots \wedge \eta_m = \det \operatorname{Coef}(\mathcal{B}) \otimes z_1 \wedge \cdots \wedge z_m$, the polynomial $\det \operatorname{Coef}(\mathcal{B})$ is semi-invariant with respect to the linear character $\det_M^{-1}$ of G for $\det_M$ the character afforded by $\wedge^m M$, as is $\det \operatorname{Coef}(\mathcal{B})_{(i)}$ likewise, and hence $h_i \in \mathcal{F}(S)^G$. By Lemma 3.7, $\det \operatorname{Coef}(\mathcal{B})$ divides $\det \operatorname{Coef}(\mathcal{B})_{(i)}$, so h_i lies in $S \cap \mathcal{F}(S)^G = S^G$ (see, e.g., [7, Section 1.7] or [21, Section 1.2]).

Remark 3.12. Of particular interest is the set $(S \otimes V^* \otimes V)^G$ of invariant differential derivations of rank 1 (see [1, 16, 17]). By Theorem 3.9, n^2 homogeneous $\mathcal{F}(S)$-independent elements in $(S \otimes V^* \otimes V)^G$ are an S^G-basis if and only if their polynomial degrees add to

$$\Delta_1 = \sum_{H \in \mathcal{A}} e_H(b_H n - b_H \delta_H + n - 1)\,.$$

Example 3.13. Consider the group

$$G = \left\langle \begin{pmatrix} 1&1&0\\0&1&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&1\\0&1&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&1&1\\0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&-1&0\\0&0&-1 \end{pmatrix} \right\rangle \subset \mathrm{GL}_3(\mathbb{F}_3)\,.$$

The subgroup W generated by the reflections of G is the group of unipotent upper triangular matrices in $\mathrm{GL}_3(\mathbb{F}_3)$ and $\mathcal{A} = \mathcal{A}(G) = \mathcal{A}(W)$ is defined by $Q = x_2^3 x_3 - x_2 x_3^3$. Three hyperplanes H in $\mathcal{A}$ each have transvection root space of dimension $b_H = 1$, whereas $b_H = 2$ for one hyperplane ($\ker x_3$). Note that $e_H = 1$ and $\delta_H = 0$ for all H in $\mathcal{A}$. The ring of W-invariant polynomials is $S^W = \mathbb{F}[f_1, f_2, f_3]$ for

$$f_1 = x_3,\;\; f_2 = x_2^3 - x_2 x_3^2,\;\; f_3 = x_1^9 - x_1^3 x_2^6 - x_1^3 x_2^4 x_3^2 - x_1^3 x_2^2 x_3^4 - x_1^3 x_3^6 + x_1 x_2^6 x_3^2 + x_1 x_2^4 x_3^4 + x_1 x_2^2 x_3^6\,.$$

Additionally, $(S \otimes V)^W$ and $(S \otimes V^*)^W$ are free S^W-modules with respective bases $\theta_1, \theta_2, \theta_3$ and $\omega_1, \omega_2, \omega_3$ by Theorem 3.2 for

$$\theta_1 = 1 \otimes v_1, \quad \theta_2 = x_1 \otimes v_1 + x_2 \otimes v_2 + x_3 \otimes v_3, \quad \theta_3 = x_1^3 \otimes v_1 + x_2^3 \otimes v_2 + x_3^3 \otimes v_3,$$
$$\omega_1 = 1 \otimes x_3, \quad \omega_2 = x_3 \otimes x_2 - x_2 \otimes x_3,$$
$$\omega_3 = (x_2^3 x_3 - x_2 x_3^3) \otimes x_1 + (-x_1^3 x_3 + x_1 x_3^3) \otimes x_2 + (x_1^3 x_2 - x_1 x_2^3) \otimes x_3\,.$$

By Theorem 3.9, $(S \otimes V^* \otimes V)^W$ is a free S^W-module with basis

$$\{d\theta_E\} \;\cup\; \{\omega_i \theta_j : 1 \le i, j \le 3\} \setminus \{\omega_3 \theta_1\}\,.$$

Here, $(x_2^3 x_3 - x_2 x_3^3)\, d\theta_E = \omega_3 \theta_1 + \omega_2 \theta_3 - x_3^2 \omega_2 \theta_2 + (x_2^3 - x_2 x_3^2) \omega_1 \theta_2$, so we may indeed replace $\omega_3 \theta_1$ (or alternatively $\omega_2 \theta_3$) by $d\theta_E$ in the set from Lemma 2.6 (with $k = 1$) to obtain a basis.

Notice that although S^G is not a polynomial ring (as G is not a reflection group, see [19]), the derivations $\theta_1, \theta_2, \theta_3$ lie in $(S \otimes V)^G$ so are a basis of the S^G-module $(S \otimes V)^G$ by Theorem 3.2. However, $(S \otimes V^*)^G$ can not be free over S^G since otherwise a basis would also serve as an S^W-basis of $(S \otimes V^*)^W$ by Theorem 3.2 (see [6, Corollary 6]) and thus would contain an element of polynomial degree 0, which is not possible as $(S_0 \otimes V^*)^G$ is empty. Note that $\omega_i \in (S \otimes V^*)^G$ only for $i = 2, 3$. Similarly, $(S \otimes V^* \otimes V)^G$ is not a free S^G-module by Theorem 3.9 as $\dim_{\mathbb{F}}(S_0 \otimes V^* \otimes V)^G$ is 1, not 2 (as for W).

4. Groups with Transvection Root Spaces Maximal

We now consider groups whose transvection root spaces are maximal, i.e., groups for which each transvection root space coincides with its reflecting hyperplane (so $b_H = n - 1$ for all H in the reflection arrangement $\mathcal{A}$). Such groups include the special and general linear groups $\mathrm{SL}_n(\mathbb{F}_q)$ and $\mathrm{GL}_n(\mathbb{F}_q)$ (see Section 9). We show that all reflecting hyperplanes are in one orbit and recall a twisted wedging that exhibits the invariant differential forms as a free exterior algebra. We also consider semi-invariant differential forms with respect to a linear character.

Only one orbit of reflecting hyperplanes. Recall that a group $G \subset \mathrm{GL}(V)$ acts on its set $\mathcal{A}$ of reflecting hyperplanes in $V = \mathbb{F}^n$ with $gH = H'$ for g in G whenever a reflection in G about $H \in \mathcal{A}$ is conjugate by g to a reflection in G about $H' \in \mathcal{A}$.

Proposition 4.1. *Let $G \subset GL(V)$ be a finite group. Any two reflecting hyperplanes of G with maximal transvection root spaces lie in the same orbit.*

Proof. Fix two such hyperplanes $H = \ker \ell_H$ and $H' = \ker \ell_{H'}$. Since the transvection root space of H is maximal, we may choose a root vector v_1 of a transvection t in G about H with $v_1 \notin H'$. Similarly, we may choose a root vector $v_2 \notin H$ of a transvection t' in G about H'. Extend v_1, v_2 to a basis $v_1, \ldots, v_n$ of V (so $v_3, \ldots, v_n$ span $H \cap H'$), and rescale v_2 and $\ell_{H'}$ so that $\ell_H(v_2) = 1$ while v_2 remains a root vector with respect to $\ell_{H'}$. Then for the dual basis $x_1, \ldots, x_n$ of V^*, $\ell_H = x_2$ and $\ell_{H'} = \alpha x_1$ for some α in $\mathbb{F}^\times$, and with this basis of V,

$$t = \left(\begin{array}{cc|ccc} 1 & 1 & & & \\ 0 & 1 & & & \\ \hline & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{array}\right) \quad \text{and} \quad t' = \left(\begin{array}{cc|ccc} 1 & 0 & & & \\ \alpha & 1 & & & \\ \hline & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{array}\right).$$

Thus $\langle t, t' \rangle$ is isomorphic to a finite subgroup of $\mathrm{SL}_2(\mathbb{F})$. We use the classification of such groups by Dickson, see [26, Chapter 3, Section 6] (see also Chapter 2, Theorem 6.8):

1) $p = 2$ and $\langle t, t' \rangle \cong D_{2m}$, the dihedral group of order $2m$, with m odd, or
2) $p = 3$ and $\langle t, t' \rangle \cong \mathrm{SL}_2(\mathbb{F}_5)$, or
3) $\langle t, t' \rangle \cong \mathrm{SL}_2(\mathbb{F}_q)$ for some p-power q,

where $p = \mathrm{char}\, \mathbb{F}$. In the first case, the transvections of $\langle t, t' \rangle$ have order 2, so correspond to the reflections in D_{2m}, which all lie in the same conjugacy class as m is odd. In the second case, the transvections have order 3, so similarly correspond to the elements of order 3 in $\mathrm{SL}_2(\mathbb{F}_5)$, which again are all in the same conjugacy class. As t and t' are conjugate in these two cases, H and H' lie in the same orbit.

In the final case, we first notice that the transvections in $K_0 := \mathrm{SL}_2(\mathbb{F}_q)$ are precisely the elements of order p, and the same is true for the group $K := \langle t, t' \rangle$ since K fixes $v_3, \ldots, v_n$. Hence the transvections t, t' in K about hyperplanes H, H', respectively, in V correspond under the isomorphism to transvections t_0, t'_0 in K_0 about some hyperplanes H_0, H'_0, respectively, in $V_0 = \mathbb{F}_q^2$. But $\mathrm{SL}_2(\mathbb{F}_q)$ acts transitively on the set of projective points in $\mathbb{F}_q^2$, i.e., all hyperplanes in V_0 are in the same K_0-orbit. Thus the pointwise stabilizer subgroups $\mathrm{Stab}_{K_0}(H_0)$ and $\mathrm{Stab}_{K_0}(H'_0)$ are conjugate in K_0, which implies that the pointwise stabilizer subgroups $\mathrm{Stab}_K(H)$ and $\mathrm{Stab}_K(H')$ are likewise conjugate in K. This follows from the fact that all of these stabilizer subgroups have a purely group-theoretic description: the transvections in K_0 about a fixed hyperplane are exactly the order p elements that commute with any fixed transvection about that hyperplane, and the same is true for K. Thus H and H' lie in the same orbit, although t and t' may not be conjugate. □

Proposition 4.1 has the following immediate implication.

Corollary 4.2. *Let $G \subset GL(V)$ be a finite group with transvection root spaces all maximal. Then G acts transitively on the set of its reflecting hyperplanes.*

Remark 4.3. For a finite group G acting linearly, when reflecting hyperplanes H, H' of G are in the same G-orbit, their pointwise stabilizers G_H, $G_{H'}$ are conjugate in G. Thus Corollary 4.2 implies that for groups G whose transvection root spaces are all maximal, we have nonnegative integers e, b, δ, and a_k such that

$$e = e_H,\ b = b_H,\ \delta = \delta_H,\ \text{and } a_k = a_{H,k} \quad \text{for all } H \in \mathcal{A}.$$

Twisted wedge product. We use the *twisted wedge product* of [13] on differential forms invariant under the action of a reflection group G whose transvection root spaces are maximal: for ω, ω' in $(S \otimes \wedge V^*)^G$, we set $\omega \curlywedge \omega' := \omega \cdot \omega'$ when rank ω or rank ω' is 0 and

$$\omega \curlywedge \omega' \ :=\ \frac{\omega \wedge \omega'}{Q^e} \quad \text{when rank } \omega, \omega' \geq 1\,, \tag{4.4}$$

for $e = e_H$ for all H in $\mathcal{A}$ (see Corollary 4.2 and Remark 4.3). Here, we use the fact that Q^e in S divides $\omega \wedge \omega'$ for all ω, ω' in $(S \otimes \wedge V^*)^G$ of rank at least 1 (see [13]). For $\omega_1, \ldots, \omega_n$ in $S \otimes V^*$ and nonempty $I = \{i_1, \ldots, i_k\} \subset [n]$ with $i_1 < \ldots < i_k$, we set

$$\omega_I^{\curlywedge} := \omega_{i_1} \curlywedge \cdots \curlywedge \omega_{i_k} = \frac{\omega_{i_1} \wedge \cdots \wedge \omega_{i_k}}{Q^{\,e(k-1)}} = \frac{\omega_I}{Q^{\,e(k-1)}} \tag{4.5}$$

and $\omega_I^{\curlywedge} = \omega_{i_1} \curlywedge \cdots \curlywedge \omega_{i_k} := 1 \otimes 1$ for $k = 0$, $I = \varnothing$.

We next use Theorem 3.2 to slightly strengthen Theorem 10 of [13], noting that the arguments in its proof hold even when S^G is not a polynomial ring. We use the free exterior algebra from [13]

$$\bigcurlywedge_{S^G}\{\omega_1, \ldots, \omega_n\} = S^G\text{-span}\{\omega_{i_1} \curlywedge \cdots \curlywedge \omega_{i_k} : 1 \leq i_1 < \ldots < i_k \leq n\} = S^G\text{-span}\{\omega_I^{\curlywedge} : I \subset [n]\}.$$

Theorem 4.6. *Let $G \subset GL(V)$ be a reflection group with transvection root spaces all maximal. Suppose $(S \otimes V^*)^G$ is a free S^G-module with basic 1-forms $\omega_1, \ldots, \omega_n$. Then $\omega_1, \ldots, \omega_n$ generate $(S \otimes \wedge V^*)^G$ as a free exterior algebra over S^G via the twisted wedge product:*

$$(S \otimes \wedge V^*)^G = \bigcurlywedge_{S^G}\{\omega_1, \ldots, \omega_n\}\,.$$

Remark 4.7. Theorem 4.6 suggests an analog of the q-*Catalan number* for groups G with $\mathrm{SL}_n(\mathbb{F}_q) \subset G \subset \mathrm{GL}_n(\mathbb{F}_q)$ (see Section 9), and more generally reflection groups $G \subset \mathrm{GL}_n(\mathbb{F})$ with maximal transvection root spaces, and exponents $m_1, \ldots, m_n$ and $S^G = \mathbb{F}[f_1, \ldots, f_n]$ for f_i homogeneous of degree d_i:

$$\mathrm{Hilb}\big((S \otimes \wedge V^*)^G, \mathsf{q}, \mathsf{t}\big) \ =\ \frac{(1 - \mathsf{q}^{e|\mathcal{A}|}) + \mathsf{q}^{e|\mathcal{A}|} \prod_{i=1}^n (1 + \mathsf{q}^{m_i - e|\mathcal{A}|}\mathsf{t})}{\prod_{i=1}^n (1 - \mathsf{q}^{d_i})}$$

is the Hilbert series of the bigraded $\mathbb{F}$-vector space of invariant differential forms,

$$\mathrm{Hilb}\big((S \otimes \wedge V^*)^G, \mathsf{q}, \mathsf{t}\big) \ :=\ \sum_{i \geq 0,\ k \geq 0} \dim_{\mathbb{F}}(S_i \otimes \wedge^k V^*)^G\ \mathsf{q}^i\, \mathsf{t}^k\,.$$

For a real reflection group G with Coxeter number h, one takes $\mathsf{t} = -\mathsf{q}^{h+1}$ to recover the q-Catalan number for G (see [2, 9]). For an extension to complex reflection groups, see [10] (and also [25]). Here one might consider $h = e|\mathcal{A}|$ (see Remark 5.7).

Semi-invariant differential forms. Let $\chi : G \to \mathbb{F}^\times$ be a linear character of a reflection group G. Then χ must be the identity on all transvections in G as they have order $p = \operatorname{char} \mathbb{F}$. This implies that Stanley's argument [23] (see also [14],[20]) for reflection groups acting over $\mathbb{C}$ extends to actions over $\mathbb{F}$ to show that

$$S^G_\chi = Q_\chi \, S^G \tag{4.8}$$

for the polynomial $Q_\chi := \prod_{H \in \mathcal{A}} \ell_H^{c_H}$, where c_H is the smallest nonnegative integer such that $\chi(s_H) = \det^{-c_H}(s_H)$ for each $H \in \mathcal{A}$. Here, s_H again is a diagonalizable reflection about H of maximal order e_H when $e_H > 1$ and the identity otherwise. Thus for $k = 0$, $(S \otimes \wedge^k V^*)^G_\chi = S^G \, (Q_\chi \otimes 1)$. We give the structure for $k > 0$ next.

Proposition 4.9. *Let $G \subset GL(V)$ be a reflection group whose transvection root spaces are all maximal. Then $(S \otimes \wedge^k V^*)^G \subset Q_{\chi^{-1}}(S \otimes \wedge^k V^*)$ for any linear character $\chi : G \to \mathbb{F}^\times$ and*

$$(S \otimes \wedge^k V^*)^G_\chi = \frac{1}{Q_{\chi^{-1}}} (S \otimes \wedge^k V^*)^G \qquad \textit{for } k > 0\,.$$

Proof. Fix $\omega \in (S \otimes \wedge^k V^*)^G$ and let $H = \ker \ell_H$ be a reflecting hyperplane of G. We use the basis $v_1, \ldots, v_n$ of V and $x_1, \ldots, x_n$ of V^* of Eq. (2.2) and write $\omega = \sum_I f_I \otimes x_I$ for some $f_I \in S$. By [13, Lemma 4], f_I is divisible by $\ell_H^{e_H}$ when $n \notin I$ since $b_H = n - 1$ and $k > 0$. Additionally, the last equation in the proof shows that f_I is divisible by $\ell_H^{e_H - 1}$ when $n \in I$. So ω lies in $\ell_H^{e_H-1} S \otimes \wedge V^*$ and, as H was arbitrary, in $Q_{\det} S \otimes \wedge V^*$. But $Q_{\chi^{-1}}$ divides $Q_{\det}$, so ω lies in $Q_{\chi^{-1}} S \otimes \wedge V^*$. This implies that $(S \otimes \wedge^k V^*)^G_\chi \supset (S \otimes \wedge^k V^*)^G / Q_{\chi^{-1}}$. The reverse inclusion follows from the fact that $Q_{\chi^{-1}} (S \otimes \wedge^k V^*)^G_\chi \subset (S \otimes \wedge^k V^*)^G$. $\square$

5. Dualizing Derivations and 1-forms

In this section, we show that $(S \otimes V)^G$ is free if and only if $(S \otimes V^*)^G$ is free over S^G for a reflection group $G \subset \mathrm{GL}(V)$ acting on $V = \mathbb{F}^n$ with transvection root spaces maximal and give a duality between exponents and coexponents. Indeed, we show how to construct dual invariant 1-forms from invariant derivations and vice versa. This in turn gives a condition for the invariant differential forms to be generated as a free exterior algebra under the twisted wedge product. We apply these results to $\mathrm{GL}_n(\mathbb{F}_q)$ and $\mathrm{SL}_n(\mathbb{F}_q)$ in Section 9.

Perfect pairing. We use a familiar perfect pairing between $\wedge^k V$ and $\wedge^{n-k} V^* \cong (\wedge^{n-k} V)^*$ related to the Hodge star operator: let Φ be the isomorphism

$$\Phi : \wedge^k V \to \wedge^{n-k} V^*, \quad \beta \ \mapsto \ \left(\beta' \mapsto \frac{\beta' \wedge \beta}{\mathrm{vol}_V} \right),$$

where $\mathrm{vol}_V = v_1 \wedge \cdots \wedge v_n$ is the volume form for a fixed choice of basis $v_1, \ldots, v_n$ of V.

Remark 5.1. Let $x_1, \ldots, x_n$ be the basis of V^* dual to the basis $v_1, \ldots, v_n$ of V. For a subset $I = \{i_1, \ldots, i_k\}$ of $[n] = \{1, \ldots, n\}$, set $I^c = [n] \backslash I$, the complementary subset. Then $\Phi(v_I) = x_{I^c}$ (see Eq. (2.4)) up to a sign given by the Levi-Civita symbol. In particular,

$$\Phi(v_1 \wedge \cdots \wedge \widehat{v_j} \wedge \cdots \wedge v_n) = (-1)^{j+1} \, x_j\,.$$

Dualizing map is semi-invariant. We extend Φ to a function $\wedge V \to \wedge V^*$. Note that Φ is a skew $\mathrm{GL}(V)$-homomorphism with respect to the character $\det = \det_V$ of $\mathrm{GL}(V)$:

Lemma 5.2. *The dualizing map Φ is $(\det^{-1})$-invariant: for any g in $GL(V)$,*

$$g(\Phi) = \det{}^{-1}(g)\,\Phi\,, \quad i.e., \;\; g\circ\Phi = \det{}^{-1}(g)\;\Phi\circ g\,.$$

Proof. Fix k and recall that $g(\mathrm{vol}_V) = \det(g)\mathrm{vol}_V$. For any $\beta \in \wedge^k V$ and $\beta' \in \wedge^{n-k} V$,

$$\big((g\Phi)(\beta)\big)(\beta') = \big(g(\Phi(g^{-1}\beta))\big)(\beta') = g\big(\Phi(g^{-1}\beta)(g^{-1}\beta')\big) = g\Big(\frac{g^{-1}\beta'\wedge g^{-1}\beta}{\mathrm{vol}_V}\Big) = \frac{\beta'\wedge\beta}{\det(g)\mathrm{vol}_V}\,,$$

which is just $\det^{-1}(g)\,\Phi(\beta)(\beta')$. □

Dual 1-forms. We use this perfect pairing to construct G-invariant 1-forms from G-invariant derivations via the linear map

$$1\otimes\Phi : S\otimes\wedge V \to S\otimes\wedge V^*\,.$$

Proposition 5.3. *Consider a reflection group $G \subset GL(V)$. Suppose $(S\otimes V)^G$ is a free S^G-module with basic derivations $\theta_1,\dots,\theta_n$. For each $1\le i\le n$, define dual 1-forms*

$$\omega_i = \theta_i^* := (1\otimes\Phi)\Big(Q_{\det}\,\theta_1\wedge\cdots\wedge\widehat{\theta}_i\wedge\cdots\wedge\theta_n\Big) \quad \in S\otimes V^*\,.$$

Then $\omega_1\dots,\omega_n$ are a basis of $(S\otimes V^*)^G$ as a free S^G-module if and only if the transvection root spaces of G are all maximal, in which case $\omega_1,\dots,\omega_n$ also generate $(S\otimes\wedge V^*)^G$ as a free exterior algebra over S^G via the twisted wedge product of Eq. (4.4).

Proof. Since G is a reflection group, $Q_{\det}$ is det-invariant (see Eq. (2.1)), and hence each ω_i is indeed invariant by Lemma 5.2 as the θ_i are invariant. We assume without loss of generality that the θ_i are homogeneous. Let A be the coefficient matrix $\mathrm{Coef}(\theta_1,\dots,\theta_n)$ so $\det A \doteq Q$ by Theorem 3.2. The determinant of the minor matrix A_{ij} is precisely the polynomial coefficient of $v_1\wedge\cdots\wedge\widehat{v}_j\wedge\cdots\wedge v_n$ in $\theta_1\wedge\cdots\wedge\widehat{\theta}_i\wedge\cdots\wedge\theta_n$. We replace each ω_i by $(-1)^{i+1}\,\omega_i$ so that the sign changes coincide with those for the cofactors c_{ij} in the cofactor matrix $C = (\det A)A^{-t}$ of A (see Remark 5.1). Then

$$\text{each}\;\; \omega_i = \sum_{j=1}^n Q_{\det}\,c_{ij}\otimes x_j \qquad \text{and} \quad \mathrm{Coef}(\omega_1,\dots,\omega_n) = Q_{\det}\,C \doteq Q\,Q_{\det}\,A^{-t}\,.$$

Thus $\det\mathrm{Coef}(\omega_1\dots,\omega_n) \doteq Q^n Q_{\det}^n \det A^{-t} \doteq Q^{n-1}\,Q_{\det}^n$, which equals $Q(\tilde{\mathcal{A}})\,Q_{\det}$ exactly when the transvection roots spaces are maximal, i.e., $b_H = n-1$ for all $H\in\mathcal{A}$. The claim then follows from Theorems 3.2 and 4.6. □

Dual derivations. Alternatively, one may use Φ^{-1} to define dual derivations. We provide a brief proof in the same style as that for Proposition 5.3.

Proposition 5.4. *Consider a reflection group $G\subset GL(V)$ for $\dim V = n > 1$ with transvection root spaces all maximal. Suppose $(S\otimes V^*)^G$ is a free S^G-module with basic 1-forms $\omega_1,\dots,\omega_n$. For each $1\le i\le n$, define dual derivations*

$$\theta_i = \omega_i^* := (1\otimes\Phi^{-1})\Big(\frac{\omega_1\curlywedge\cdots\curlywedge\widehat{\omega}_i\curlywedge\cdots\curlywedge\omega_n}{Q_{\det}}\Big)\,.$$

Then $\theta_1,\dots,\theta_n$ are a basis of $(S\otimes V)^G$ as a free S^G-module.

Proof. By Proposition 4.9, $Q_{\det}$ divides in $S\otimes\wedge V^*$ the indicated twisted wedge product and the quotient is $(\det^{-1})$-invariant. By Lemma 5.2, the map Φ^{-1} is itself det-invariant and thus each θ_i lies $(S\otimes V)^G$. Consider the coefficient matrix $A=\mathrm{Coef}(\omega_1,\dots,\omega_n)$ and the cofactor matrix $C=\{c_{ij}\}=(\det A)A^{-t}$ of A. We assume without loss of generality that the ω_i are homogeneous. Then $\det A\doteq Q^{en-1}$ by Theorem 3.2 (as $b_H=n-1$ and $e=e_H$ for all $H\in\mathcal{A}$) and, after replacing θ_i by $(-1)^{i+1}\,\theta_i$ to match the sign changes of C,

$$\text{each}\quad \theta_i=\sum_{j=1}^{n}\frac{c_{ij}}{Q^{e(n-2)}\,Q_{\det}}\qquad\text{and}\qquad \mathrm{Coef}(\theta_1,\dots,\theta_n)=\frac{C}{Q^{e(n-2)}\,Q_{\det}}\doteq Q^{e}A^{-t}\,.$$

The claim follows from Theorem 3.2 since $\det\mathrm{Coef}(\theta_1,\dots,\theta_n)=Q^{en}\det A^{-t}\doteq Q$. $\square$

Remark 5.5. A version of Proposition 5.4 also holds when $n=1$ provided we again apply $(1\otimes\Phi^{-1})$ to a generator of $(S\otimes\wedge^{n-1}V^*)^G_{\det^{-1}}$ (see Proposition 4.9). For $n=1$, we use the bases v of V and x of V^* as well as the derivation θ and 1-form ω from Example 3.10. Here, $(S\otimes\wedge^{n-1}V^*)^G_{\det^{-1}}=S^G(Q\otimes 1)$ since $Q=Q_{\det^{-1}}=x$, see Eq. (2.1) or Eq. (4.8), so we apply $(1\otimes\Phi^{-1})$ to $Q\otimes 1$ to dualize ω:

$$\omega^*=(1\otimes\Phi^{-1})(Q\otimes 1)=Q\otimes v=x\otimes v=\theta\,.$$

Note that the dual of θ here by Proposition 5.3 is just

$$\theta^*=(1\otimes\Phi)(Q_{\det}\otimes 1)=Q_{\det}\otimes x=\omega\qquad\text{for } Q_{\det}=x^{e-1}\,.$$

Duality of exponents and coexponents. Propositions 5.3 and 5.4 imply an analog of the duality of *exponents* and *coexponents* (see [15]) for well-generated complex reflection groups. Recall that for any finite group G, if $(S\otimes V^*)^G$ is a free S^G-module, the set of polynomial degrees in a homogeneous basis does not depend on choice of basis, and likewise for $(S\otimes V)^G$.

Corollary 5.6. *Let $G\subset GL(V)$ be a reflection group with transvection root spaces all maximal. Then $(S\otimes V)^G$ is a free S^G-module if and only if $(S\otimes V^*)^G$ is a free S^G-module. When both modules are free with respective homogeneous bases of polynomial degrees $m_1^*,\dots,m_n^*$ and $m_1,\dots,m_n$, then*

$$m_i^*+m_i=e|\mathcal{A}|\,,$$

after possibly reindexing, where $|\mathcal{A}|$ is the number of reflecting hyperplanes of G and e is the maximal order of a diagonalizable reflection in G.

Proof. Suppose $\theta_1,\dots,\theta_n$ are a homogeneous S^G-basis of $(S\otimes V)^G$. Then the dual 1-forms $\omega_1,\dots,\omega_n$ afforded by Proposition 5.3 give an S^G-basis of $(S\otimes V^*)^G$ with

$$m_i=\deg\omega_i=\deg Q_{\det}+\sum_{j\neq i}\deg\theta_j=\deg Q_{\det}+\deg Q-\deg\theta_i=e\deg Q-m_i^*=e|\mathcal{A}|-m_i^*\,,$$

since ω_i is dual to θ_i and $Q_{\det}Q=Q^e$. Alternatively, if the 1-forms $\omega_1,\dots,\omega_n$ are an S^G-basis of $(S\otimes V^*)^G$, then the dual derivations $\theta_1,\dots,\theta_n$ are an S^G-basis for $(S\otimes V)^G$ by Proposition 5.4 (use Remark 5.5 when $n=1$) with again $m_i+m_i^*=e|\mathcal{A}|$. $\square$

Remark 5.7. Recall that the exponents m_i and coexponents m_i^* of a *duality (well-generated) complex reflection group* $G \subset \mathrm{GL}_n(\mathbb{C})$ (e.g. any Weyl or Coxeter group) may be ordered so

$$m_i + m_i^* = \text{Coxeter number} = \deg f_n = \frac{\deg QJ}{n} = \frac{\deg QQ_{\det}Q(\tilde{\mathcal{A}})}{n}$$

as $\sum_i m_i^* = \deg Q$ and $\sum_i m_i = \deg J$ for the Jacobian determinant $J = \det\{\frac{\partial f_i}{\partial x_j}\} \doteq Q_{\det} = Q_{\det}Q(\tilde{\mathcal{A}})$ as G contains no transvections. Here, $f_1, \dots, f_n$ are homogeneous basic invariants for G ordered with nondecreasing degrees. One thus may be tempted by Corollary 5.6 to regard the integer

$$e|\mathcal{A}| = (\text{maximal order of a diagonalizable reflection}) \cdot (\#\text{reflecting hyperplanes})$$

as the *Coxeter number* of a reflection group $G \subset \mathrm{GL}_n(\mathbb{F})$ for arbitrary $\mathbb{F}$ with transvection root spaces all maximal. In this case, we use $QQ_{\det}Q(\tilde{\mathcal{A}})$ in favor of the *discriminant* QJ and note that

$$e|\mathcal{A}| = \frac{ne|\mathcal{A}|}{n} = \frac{|\mathcal{A}| + (e-1)|\mathcal{A}| + (n-1)e|\mathcal{A}|}{n} = \frac{\deg QQ_{\det}Q(\tilde{\mathcal{A}})}{n}.$$

Reflection groups with transvection root spaces all maximal thus may serve as modular analogues of the duality (well-generated) complex reflection groups (also see Section 9).

6. Structure Theorem for Invariant Differential Derivations

We investigate the structure of $(S \otimes \wedge V^* \otimes V)^G$ when $G \subset \mathrm{GL}(V)$ is a reflection group with transvection root spaces maximal. Such is the case when $\mathrm{SL}_n(\mathbb{F}_q) \subset G \subset \mathrm{GL}_n(\mathbb{F}_q)$ (see Section 9) or G is the pointwise stabilizer in $\mathrm{SL}_n(\mathbb{F}_q)$ or $\mathrm{GL}_n(\mathbb{F}_q)$ of a hyperplane in V, for example. Recall that $(S \otimes V)^G$ is free if and only if $(S \otimes V^*)^G$ is free over S^G in this setting (see Corollary 5.6). We start with basic derivations in $(S \otimes V)^G$ and use the dual 1-forms in $(S \otimes V^*)^G$ afforded by Proposition 5.3 to construct an S^G-basis for $(S \otimes \wedge V^* \otimes V)^G$. Alternatively, we could instead construct the same S^G-basis for $(S \otimes \wedge V^* \otimes V)^G$ starting with basic 1-forms in $(S \otimes V^*)^G$ and using the dual derivations in $(S \otimes V)^G$ from Proposition 5.4.

We suppose $\operatorname{char} \mathbb{F} \neq 2$ in this section and save the $\operatorname{char} \mathbb{F} = 2$ case for Section 7. Recall that $V = \mathbb{F}^n$ and we write $I \in \binom{[n]}{k}$ when $I \subset [n] = \{1, \dots, n\}$ with $|I| = k$.

Lemma 6.1. *Consider a reflection group $G \subset GL(V)$ with transvection root spaces all maximal. Suppose $(S \otimes V)^G$ is a free S^G-module with basic derivations $\theta_1, \dots, \theta_n$ and dual 1-forms $\omega_1, \dots, \omega_n$. Then the following two subsets of $(S \otimes \wedge V^* \otimes V)^G$ of size $m = n\binom{n}{k}$ are $\mathcal{F}(S)$-independent for any $r = 1, \dots, n$:*

- $\{d\theta_E\} \cup \{\omega_I^{\curlywedge}\, \theta_1, \dots, \omega_I^{\curlywedge}\, \theta_n : I \subset [n]\} \setminus \{\omega_r \theta_r\}$
- $\{\omega_I^{\curlywedge}\, \theta_1, \dots, \omega_I^{\curlywedge}\, \theta_{r-1}, \omega_I^{\curlywedge}\, \theta_{r+1}, \dots, \omega_I^{\curlywedge}\, \theta_n : I \subset [n],\, r \in I\} \cup \{\omega_I^{\curlywedge}\, d\theta_E, \omega_I^{\curlywedge}\, \theta_1, \dots, \omega_I^{\curlywedge}\, \theta_n\colon I \subset [n], r \notin I\}.$

Proof. Fix r and note that the given forms indeed all lie in $(S \otimes \wedge V^* \otimes V)^G$ (see Eq. (4.4) and Proposition 5.3). For each k, denote the collection of elements of rank k in the first set in the claim by $\mathcal{B}_k$ and in the second set by $\mathcal{B}'_k$. When $k = 0$, $\mathcal{B}_k = \mathcal{B}'_k = \{\theta_1, \dots, \theta_n\}$, which is independent over $\mathcal{F}(S)$ as it is a basis of $(S \otimes V)^G$. Note that Lemma 2.6 implies that $\mathcal{C}_k = \{\omega_I^{\curlywedge}\, \theta_j : I \in \binom{[n]}{k},\, 1 \le j \le n\}$ is independent over $\mathcal{F}(S)$ for all k. As $\mathcal{B}_k = \mathcal{C}_k$ for $k \ge 2$ and $\mathcal{B}_1 = \mathcal{B}'_1$, it is left to show that, for $k \ge 1$,

$$\mathcal{B}'_k = \left\{\omega_I^{\curlywedge}\, \theta_j : I \in \tbinom{[n]}{k} \text{ with } r \notin I \text{ or } j \neq r\right\} \cup \left\{\omega_I^{\curlywedge}\, d\theta_E : I \in \tbinom{[n]}{k-1} \text{ with } r \notin I\right\}$$

is independent over $\mathcal{F}(S)$. Recall from the proof of Proposition 5.3 that

$$\mathrm{Coef}(\omega_1,\dots,\omega_n) = Q_{\det}\, C \doteq Q\, Q_{\det}\, A^{-t} \quad \text{ for } \mathrm{Coef}(\theta_1,\dots,\theta_n) = A,$$

after replacing ω_i by $(-1)^{i+1}\omega_i$, where $C = \{c_{ij}\} = (\det A)\, A^{-t}$ is the cofactor matrix of $A = \{a_{ij}\}$, as $\det A \doteq Q$ by Theorem 3.2. Then $\omega_m = Q_{\det} \sum_i c_{mi} \otimes x_i$ and $\theta_m = \sum_j a_{mj} \otimes v_j$ for $1 \le m \le n$ and, as $C^t A = (\det A)\, I$,

$$\sum_{m=1}^{n} \omega_m\, \theta_m = Q_{\det} \sum_{i,j} \Big(\sum_{m=1}^{n} c_{mi} a_{mj} \Big) \otimes x_i \otimes v_j \doteq Q_{\det} \sum_i Q \otimes x_i \otimes v_i = Q^e\, d\theta_E\,.$$

Hence, for any $I \subset \{1,\dots,n\}$,

$$\omega_I^{\curlywedge}\, d\theta_E \doteq \frac{1}{Q^e} \sum_{m=1}^{n} (\omega_I^{\curlywedge} \wedge \omega_m)\, \theta_m = \sum_{m \notin I} \Big(\frac{\omega_I^{\curlywedge} \wedge \omega_m}{Q^e} \Big)\, \theta_m\,.$$

Thus each $\omega_I^{\curlywedge}\, d\theta_E$ lies in the $\mathcal{F}(S)$-span of $\mathcal{C}_k$ with nonzero coefficient of $\omega_{I\cup\{r\}}^{\curlywedge}\theta_r$ when $r \notin I$. As the various sets $I \cup \{r\}$ with $r \notin I$ are distinct, $\mathcal{B}'_k$ is $\mathcal{F}(S)$-independent for $k \ge 1$. □

Now that we have $\mathcal{F}(S)$-independent sets of the appropriate size, we show that they each yield a basis of $(S \otimes \wedge V^* \otimes V)^G$. We obtain Theorem 1.2 of the introduction, again using the dual 1-forms of Proposition 5.3:

Theorem 6.2. *Let $G \subset GL(V)$ be a reflection group with transvection root spaces all maximal and* $\operatorname{char} \mathbb{F} \neq 2$. *Suppose $(S \otimes V)^G$ is a free S^G-module with basic derivations $\theta_1,\dots,\theta_n$ and dual 1-forms $\omega_1,\dots,\omega_n$. Then $(S \otimes \wedge V^* \otimes V)^G$ is a free S^G-module with basis*

$$\big\{d\theta_E\big\} \cup \big\{\omega_I^{\curlywedge}\,\theta_1,\dots,\omega_I^{\curlywedge}\,\theta_n : I \subset [n]\big\} \setminus \big\{\omega_r\theta_r\big\} \quad \textit{for any } r = 1,\dots,n.$$

Proof. Fix r and assume without loss of generality that the θ_i are homogeneous. For each k, let $\mathcal{B}_k$ be the collection of elements in the proposed basis of rank k. Then $\mathcal{B}_k$ is an $\mathcal{F}(S)$-independent subset of $(S \otimes \wedge^k V^* \otimes V)^G$ by Lemma 6.1. It suffices to show by Theorem 3.9 that $\sum_{\eta\in\mathcal{B}_k} \deg\eta = \Delta_k$ for

$$\Delta_k = \begin{cases} \deg Q & \text{when } k = 0\,, \\ (en^2 - e)\deg Q & \text{when } k = 1\,, \\ \binom{n}{k}(en - k + 1)\deg Q & \text{when } k \ge 2\,. \end{cases}$$

When $k = 0$, $\mathcal{B}_k = \{\theta_1,\dots,\theta_n\}$, so indeed $\sum_{\eta\in\mathcal{B}_k} \deg\eta = \deg Q = \Delta_0$ by Theorem 3.2.

When $k = 1$, $\mathcal{B}_k = \{d\theta_E\} \cup \{\omega_i\theta_j : (i,j) \neq (r,r)\}$, thus

$$\sum_{\eta\in\mathcal{B}_1} \deg\eta = \sum_{1\le i,j\le n} \deg\omega_i\theta_j - \deg\omega_r\theta_r + \deg d\theta_E = n\sum_i \deg\omega_i + n\sum_j \deg\theta_j - \deg\omega_r\theta_r + \deg d\theta_E,$$

which is $(en^2 - e)\deg Q = \Delta_1$ since $\sum_i \deg\omega_i = (en-1)\deg Q$ and $\sum_j \deg\theta_j = \deg Q$ by Theorem 3.2 (see Proposition 5.3), $\deg d\theta_E = 0$, and (see the proof of Corollary 5.6)

$$\deg\omega_r\theta_r = \deg\omega_r + \deg\theta_r = e|\mathcal{A}| = e\deg Q\,.$$

When $k \ge 2$,

$$\sum_{\eta\in\mathcal{B}_k} \deg\eta = \sum_{I\in\binom{[n]}{k},\, 1\le j\le n} \deg\omega_I^{\curlywedge}\,\theta_j = n\sum_{I\in\binom{[n]}{k}} \deg\omega_I^{\curlywedge} + \binom{n}{k}\sum_j \deg\theta_j\,.$$

As $\deg \omega_I^{\curlywedge} = \sum_{i\in I} \deg \omega_i - e(k-1)\deg Q$ (see Eq. (4.4)) and $n\binom{n-1}{k-1} = k\binom{n}{k}$, this is

$$n\tbinom{n-1}{k-1}\sum_i \deg \omega_i - n\tbinom{n}{k}e(k-1)\deg Q + \tbinom{n}{k}\sum_j \deg\theta_j = \tbinom{n}{k}(en-k+1)\deg Q = \Delta_k\,. \qquad \square$$

Remark 6.3. Theorem 6.2 implies that $\{\omega_I^{\curlywedge}\,\theta_j : I \in \binom{[n]}{k}, 1 \le j \le n\}$ is an S^G-basis of $(S\otimes\wedge^k V^*\otimes V)^G$ when $k \ge 2$ for a reflection group $G \subset \mathrm{GL}(V)$ with transvection root spaces all maximal and $\operatorname{char}\mathbb{F} \neq 2$, provided $(S\otimes V)^G$ is free with basic derivations $\theta_1,\ldots,\theta_n$ and dual 1-forms $\omega_1,\ldots,\omega_n$.

We see in Section 7 that the following corollary of Theorem 6.2 also holds when $\operatorname{char}\mathbb{F} = 2$.

Corollary 6.4. *Suppose* $\operatorname{char}\mathbb{F} \neq 2$. *Let $G \subset GL(V)$ be a reflection group with transvection root spaces all maximal. If $(S\otimes V)^G$ is a free S^G-module, then so is $(S\otimes\wedge V^*\otimes V)^G$.*

We give an alternate basis from which we derive a module structure in Corollary 6.7, again using the dual 1-forms of Proposition 5.3.

Theorem 6.5. *Let $G \subset GL(V)$ be a reflection group with transvection root spaces all maximal and* $\operatorname{char}\mathbb{F} \neq 2$. *Suppose $(S\otimes V)^G$ is a free S^G-module with basic derivations $\theta_1,\ldots,\theta_n$ and dual 1-forms $\omega_1,\ldots,\omega_n$. Then $(S\otimes\wedge V^*\otimes V)^G$ is a free S^G-module with basis*

$$\left\{\omega_I^{\curlywedge}\,\theta_1,\ldots,\omega_I^{\curlywedge}\,\theta_{r-1},\omega_I^{\curlywedge}\,\theta_{r+1},\ldots,\omega_I^{\curlywedge}\,\theta_n : I\subset[n],\, r\in I\right\}\cup\left\{\omega_I^{\curlywedge}\,d\theta_E,\omega_I^{\curlywedge}\,\theta_1,\ldots,\omega_I^{\curlywedge}\,\theta_n : I\subset[n], r\notin I\right\}$$

for any $r = 1,\ldots,n$.

Proof. Fix r and assume without loss of generality that the θ_i are homogeneous. For each k, let $\mathcal{B}'_k$ be the collection of elements in the proposed basis of rank k. Then $\mathcal{B}'_k$ is an $\mathcal{F}(S)$-independent subset of $(S\otimes\wedge^k V^*\otimes V)^G$ by Lemma 6.1. We argue that $\sum_{\eta\in\mathcal{B}'_k}\deg\eta$ coincides with $\sum_{\eta\in\mathcal{B}_k}\deg\eta$, where $\mathcal{B}_k$ is the collection of elements in the basis afforded by Theorem 6.2 of rank k for this fixed r. It will follow then from Theorem 3.9 that $\mathcal{B}'_k$ is also a basis.

Notice $\mathcal{B}_k = \mathcal{B}'_k$ for $k = 0, 1$, so assume $k \ge 2$. Recall that $\deg\omega_r\theta_r = e\deg Q$ (see the proof of Theorem 6.2). Then for nonempty $I \subset \{1,\ldots,n\}$ with $r \notin I$,

$$\deg\omega_{I\cup\{r\}}\theta_r = \deg\omega_I^{\curlywedge}\curlywedge\omega_r\,\theta_r = \deg\omega_I^{\curlywedge} + \deg\omega_r\theta_r - e\deg Q = \deg\omega_I^{\curlywedge} = \deg\omega_I^{\curlywedge}\,d\theta_E\,.$$

Thus, as $k \ge 2$, the elements in $\mathcal{B}_k$ not in $\mathcal{B}'_k$ have the same polynomials degrees as the elements in $\mathcal{B}'_k$ not in $\mathcal{B}_k$, and thus $\sum_{\eta\in\mathcal{B}'_k}\deg\eta = \sum_{\eta\in\mathcal{B}_k}\deg\eta$. $\square$

Remark 6.6. Recall again that when the transvection root spaces of a reflection group G are all maximal, $(S\otimes V)^G$ is free over S^G if and only if $(S\otimes V^*)^G$ is free over S^G (see Corollary 5.6). For Theorem 6.2 and Theorem 6.5, rather than assuming that $(S\otimes V)^G$ is free and using the dual 1-forms, we may instead assume $(S\otimes V^*)^G$ is free and use the dual derivations of Proposition 5.4 (see Remark 5.5 for $n = 1$) to obtain the same S^G-bases of $(S\otimes\wedge V^*\otimes V)^G$. Indeed, the proofs simply rely on the fact that ω_i and θ_i are dual.

Module structure over twisted subalgebra. For a well-generated complex reflection group G, $(S\otimes\wedge V^*\otimes V)^G$ is a direct sum of submodules of rank 1 over

$$\bigwedge_{S^G}\{\omega_1,\ldots,\omega_{n-1}\}$$

for homogeneous basic 1-forms $\omega_1,\ldots,\omega_n$ with ω_n of maximal polynomial degree (see [16, Theorem 1.1]). One asks if a similar result holds over arbitrary fields $\mathbb{F}$ for reflection groups whose transvection root spaces are maximal. Theorem 6.5 implies a more subtle decomposition with three key differences from the characteristic zero setting: here we may omit any one of the basic 1-forms in constructing a suitable subalgebra of invariant differential forms, we use the twisted wedge product of Eq. (4.5) instead of the regular wedge product, and we require an ideal of invariant differential forms. We define for $r = 1, \dots, n$

$$A_r := \bigcurlywedge_{S^G}\{\omega_1, \dots, \widehat{\omega_r}, \dots, \omega_n\} = S^G\text{-span}\{\omega_I^{\curlywedge} : I \subset [n], r \notin I\} \quad \subset (S \otimes \wedge V^*)^G$$

and use the ideal generated by ω_r under $\curlywedge$:

$$A_r \curlywedge \omega_r := S^G\text{-span}\{\omega_I^{\curlywedge} : I \subset [n] \text{ with } r \in I\}\,.$$

The following corollary of Theorem 6.5 provides a module structure over A_r.

Corollary 6.7. *Let $G \subset GL(V)$ be a reflection group with transvection root spaces all maximal and $\operatorname{char}\mathbb{F} \neq 2$. Suppose $(S \otimes V)^G$ is a free S^G-module with basic derivations $\theta_1, \dots, \theta_n$ and dual 1-forms $\omega_1, \dots, \omega_n$. Then, for any $r = 1, \dots, n$, $(S \otimes \wedge V^* \otimes V)^G$ is a direct sum of A_r-submodules:*

$$(S \otimes \wedge V^* \otimes V)^G = \bigoplus_{j=1}^{n} A_r\, \theta_j \oplus \bigoplus_{j \neq r} (A_r \curlywedge \omega_r)\, \theta_j \oplus A_r\, d\theta_E\,.$$

Hilbert series. We consider the Hilbert series of the bigraded $\mathbb{F}$-vector space of invariant differential derivations:

$$\operatorname{Hilb}\big((S \otimes \wedge V^* \otimes V)^G, \mathsf{q}, \mathsf{t}\big) \;:=\; \sum_{i \geq 0,\ k \geq 0} \dim_{\mathbb{F}}(S_i \otimes \wedge^k V^* \otimes V)^G\ \mathsf{q}^i\, \mathsf{t}^k\,.$$

For a Coxeter group G, this Hilbert series gives the first Kirkman number (see [18, 1, 2, 17]).

Corollary 6.8. *Let $G \subset GL(V)$ be a reflection group with transvection root spaces all maximal and $\operatorname{char}\mathbb{F} \neq 2$. If $(S \otimes V)^G$ is a free S^G-module with homogeneous generators of polynomial degrees $m_1^*, \dots, m_n^*$, then*

$$\begin{aligned}&\operatorname{Hilb}\big((S \otimes \wedge V^* \otimes V)^G, \mathsf{q}, \mathsf{t}\big)\\ &\quad= \operatorname{Hilb}\big(S^G, \mathsf{q}\big)\Big(\mathsf{t} - \mathsf{q}^{e|\mathcal{A}|}\mathsf{t} + \big(\sum_{i=1}^{n} \mathsf{q}^{m_i^*}\big)\big(1 - \mathsf{q}^{e|\mathcal{A}|} + \mathsf{q}^{e|\mathcal{A}|}\prod_{i=1}^{n}(1 + \mathsf{q}^{-m_i^*}\mathsf{t})\big)\Big)\,,\end{aligned}$$

where e is the maximal order of a diagonalizable reflection in G and $|\mathcal{A}|$ is the number of reflecting hyperplanes of G.

Proof. Say $\theta_1, \dots, \theta_n$ is a homogeneous S^G-basis of $(S \otimes V)^G$ with $\deg\theta_i = m_i^*$. Consider the dual 1-forms $\omega_1, \dots, \omega_n$ of polynomial degree $m_i = \deg\omega_i = e|\mathcal{A}| - m_i^*$ of Proposition 5.3 (see Corollary 5.6) and set $A = \curlywedge_{\mathbb{F}}\{\omega_1, \dots, \omega_n\} = \mathbb{F}\text{-span}\{\omega_I^{\curlywedge} : I \subset [n]\}$. Then Theorem 6.2 (with $r = n$ for example) implies the result: $\operatorname{Hilb}\big((S \otimes \wedge V^* \otimes V)^G, \mathsf{q}, \mathsf{t}\big)$ is $\operatorname{Hilb}\big(S^G, \mathsf{q}\big)$ times

$$\operatorname{Hilb}\Big(\bigoplus_{I,j} \mathbb{F}\,\omega_I^{\curlywedge}\,\theta_j, \mathsf{q}, \mathsf{t}\Big) - \operatorname{Hilb}\Big(\mathbb{F}\,\omega_n\,\theta_n, \mathsf{q}, \mathsf{t}\Big) + \operatorname{Hilb}(\mathbb{F}\,d\theta_E, \mathsf{q}, \mathsf{t})\,,$$

and a computation confirms that $\operatorname{Hilb}\big(\bigoplus_{I,j} \mathbb{F}\,\omega_I^{\curlywedge}\,\theta_j, \mathsf{q}, \mathsf{t}\big)$ is

$$\operatorname{Hilb}(A, \mathsf{q}, \mathsf{t}) \cdot \operatorname{Hilb}(\oplus_j\ \mathbb{F}\,\theta_j, \mathsf{q}, \mathsf{t}) = \Big(1 - \mathsf{q}^{e|\mathcal{A}|} + \mathsf{q}^{e|\mathcal{A}|}\prod_{i=1}^{n}(1 + \mathsf{q}^{m_i - e|\mathcal{A}|}\,\mathsf{t})\Big)\Big(\sum_{i=1}^{n} \mathsf{q}^{m_i^*}\Big)\,.$$

□

See Section 9 for examples of reflection groups with transvection root spaces maximal. Groups fixing a single hyperplane pointwise provide other examples, see Proposition 3.8 (with $b_H = n-1$) and Eq. (2.2). Note that the hypothesis that the transvection root spaces of G are all maximal in Theorem 6.2 is critical, as we see in the following example.

Example 6.9. We consider a reflection group G over $\mathbb{F}_p$ for a prime $p > 2$ where S^G is not a polynomial algebra (see [7, Section 8.2]):

$$G = \left\langle \begin{pmatrix} 1&0&0&0\\0&1&0&0\\1&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&1&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0\\0&1&0&0\\1&1&1&0\\1&1&0&1 \end{pmatrix} \right\rangle = \left\{ \begin{pmatrix} 1&0&0&0\\0&1&0&0\\a&c&1&0\\c&b&0&1 \end{pmatrix} : a,b,c \in \mathbb{F}_p \right\}.$$

Here, $Q = x_1^p x_2 - x_1 x_2^p$ and $e_H = b_H = 1$, and $\delta_H = 0$ for each reflecting hyperplane H of G. Both $(S \otimes V)^G$ and $(S \otimes V^*)^G$ are free S^G-modules with respective bases $\{\theta_i\}$ and $\{\omega_i\}$ for

$$\begin{aligned}
\theta_1 &= 1 \otimes v_3, & \theta_3 &= x_1 \otimes v_1 + x_2 \otimes v_2 + x_3 \otimes v_3 + x_4 \otimes v_4,\\
\theta_2 &= 1 \otimes v_4, & \theta_4 &= x_1^p \otimes v_1 + x_2^p \otimes v_2 + x_3^p \otimes v_3 + x_4^p \otimes v_4,\\
\omega_1 &= 1 \otimes x_1, & \omega_3 &= x_3 \otimes x_1 + x_4 \otimes x_2 - x_1 \otimes x_3 - x_2 \otimes x_4,\\
\omega_2 &= 1 \otimes x_2, & \omega_4 &= x_3^p \otimes x_1 + x_4^p \otimes x_2 - x_1^p \otimes x_3 - x_2^p \otimes x_4\,.
\end{aligned}$$

None of the transvection root spaces are maximal, and the conclusion of Theorem 6.2 fails:

$$\{d\theta_E\} \cup \{\omega_i\theta_j : (i,j) \neq (a,b)\} \text{ is } \textit{not} \text{ an } S^G\text{-basis of } (S \otimes V^* \otimes V)^G \text{ for any } (a,b).$$

Remark 6.10. The arguments of Sections 5 and 6 apply when $\operatorname{char}\mathbb{F} \neq 2$ to any finite group G with transvection root spaces maximal if $\det(g)^e = 1$ for every $g \in G$, where e is the maximal order of a diagonalizable reflection in G. Indeed, in this case, $Q_{\det}$ is det-invariant and the arguments in the proofs of Sections 5 and 6 show that $(S \otimes \wedge V^* \otimes V)^G$ is free over S^G when $(S \otimes V)^G$ is free over S^G, with explicit basis given by Theorem 6.2 or Theorem 6.5. However, we have yet to even find an example of a nonreflection group whose transvection root spaces are all maximal with $(S \otimes V)^G$ free over S^G.

7. The Case of Characteristic 2

In this section, we consider the case when $\operatorname{char}\mathbb{F}$ is 2 and G is a reflection group with transvection root spaces maximal. Examples include $\mathrm{SL}_n(\mathbb{F}_q)$ and $\mathrm{GL}_n(\mathbb{F}_q)$ for q a power of 2. The structure of $(S \otimes \wedge V^* \otimes V)^G$ may differ from that in Section 6 where $\operatorname{char}\mathbb{F} \neq 2$. Indeed, in Appendix A, we must distinguish the groups whose pointwise stabilizers of hyperplanes consist of exactly one transvection and the identity element, i.e., $\delta_H = 1$ for all H in the reflection arrangement $\mathcal{A}$, which only occurs when $\operatorname{char}\mathbb{F} = 2$ (see Eq. (3.3)).

Theorem 7.1. *Let $G \subset GL(V)$ be a reflection group with transvection root spaces all maximal and* $\operatorname{char}\mathbb{F} = 2$.

- *If $|\mathcal{A}| = 1$, $(S \otimes V)^G$, $(S \otimes V^*)^G$, and $(S \otimes \wedge V^* \otimes V)^G$ are free S^G-modules with structure given by Lemma 3.1 and Proposition 3.8.*
- *If $|\mathcal{A}| \neq 1$ and G_H comprises a single transvection and the identity for each $H \in \mathcal{A}$, then $(S \otimes V)^G$, $(S \otimes V^*)^G$, and $(S \otimes \wedge V^* \otimes V)^G$ are free S^G-modules. In this case, $n = 2$ and $(S \otimes \wedge V^* \otimes V)^G$ has rank 4 over an S^G-submodule of $(S \otimes \wedge V^*)^G$ of rank 2.*
- *Otherwise, $(S \otimes \wedge V^* \otimes V)^G$ is a free S^G-module, provided $(S \otimes V)^G$ is a free S^G-module, with bases given by Theorems 6.2 and 6.5 and module structure given by Corollary 6.7.*

Proof. For the first claim, we just appeal to Lemma 3.1 and Proposition 3.8. In the case of the third claim, $\delta_H = 0$ for some $H \in \mathcal{A}$ (see Eq. (3.3)) so in fact $\delta_H = 0$ for all $H \in \mathcal{A}$ by

Corollary 4.2 (see Remark 4.3). Then the arguments in the proofs of Theorems 6.2 and 6.5 and Corollary 6.7 hold.

Now assume we are in the setting of the second claim. Then $\delta_H = 1$ for every $H \in \mathcal{A}$ and there are nonnegative integers e, b, and a_k such that $e = e_H$, $b = b_H$, and $a_k = a_{H,k}$ for all $H \in \mathcal{A}$ (again see Remark 4.3). In this case, G contains no diagonalizable reflections and $e = 1$. Further, each transvection root space of G has dimension $b = 1$ as it is spanned by a single transvection. But this forces $\dim V = n = 2$ as each transvection root space is also maximal, and thus G is a finite subgroup of $\mathrm{SL}_2(\mathbb{F})$ as G is generated by its transvections. Then G must be isomorphic (as an abstract group) to some dihedral group D_{2m} of order $2m$ with m odd by the classification of Dickson (see [26, Chapter 3, Section 6]). There are exactly m elements of order 2 in D_{2m} as m is odd, and thus $|\mathcal{A}| = m$ is odd as the transvections are the only elements of order 2 in G and there is only one transvection per hyperplane. Hence $\deg Q$ is greater than 2 and is odd as $|\mathcal{A}| \neq 1$.

As G is generated by two transvections, there is some $\alpha \in \mathbb{F}^\times$ and a basis v_1, v_2 of V with dual basis x_1, x_2 of V^* so that $G = \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ \alpha & 1 \end{smallmatrix}\right) \rangle$. The following derivations θ_1, θ_2 are an S^G-basis of $(S \otimes V)^G$ (see, e.g., [15, Section B.2]), and their dual 1-forms ω_1, ω_2 are an S^G-basis of $(S \otimes V^*)^G$ by Proposition 5.3:

$$\begin{aligned} \theta_1 &= \tfrac{\partial Q}{\partial x_2} \otimes v_1 + \tfrac{\partial Q}{\partial x_1} \otimes v_2\,, \quad & \theta_2 &= x_1 \otimes v_1 + x_2 \otimes v_2\,, \\ \omega_1 &= x_2 \otimes x_1 + x_1 \otimes x_2\,, & \omega_2 &= \tfrac{\partial Q}{\partial x_1} \otimes x_1 + \tfrac{\partial Q}{\partial x_2} \otimes x_2\,. \end{aligned}$$

We now give an explicit S^G-basis for each $(S \otimes \wedge^k V^* \otimes V)^G$ using Theorem 3.9.

For $k = 0$, the derivations θ_1, θ_2 are an S^G-basis as they are $\mathcal{F}(S)$-independent and have polynomial degrees summing to $\deg Q$.

For $k = 1$, we argue that the four forms $\omega_1\theta_1, \omega_1\theta_2, d\theta_E, \eta_0$ are $\mathcal{F}(S)$-independent, lie in $(S \otimes V^* \otimes V)^G$, and have polynomial degrees summing to $2\deg Q$ for

$$\eta_0 = Q^{-1}(\omega_2\theta_1 + f^{\deg Q - 2}\omega_1\theta_2) \quad \text{with} \quad f = x_1^2 + x_1x_2 + \alpha^{-1}x_2^2 \ \in S^G\,.$$

First notice that the forms $\omega_1\theta_1, \omega_1\theta_2, \omega_2\theta_1, \omega_2\theta_2$ are $\mathcal{F}(S)$-independent by Lemma 2.6, which implies that $\omega_1\theta_1, \omega_1\theta_2, \omega_2\theta_1, d\theta_E$ are also $\mathcal{F}(S)$-independent since

$$\omega_1\theta_1 + \omega_2\theta_2 = (x_1\tfrac{\partial Q}{\partial x_1} + x_2\tfrac{\partial Q}{\partial x_2}) \otimes x_1 \otimes v_1 + (x_1\tfrac{\partial Q}{\partial x_1} + x_2\tfrac{\partial Q}{\partial x_2}) \otimes x_2 \otimes v_2 = (\deg Q)\, Q\, d\theta_E \neq 0$$

as $\deg Q = |\mathcal{A}|$ is odd and Q is homogeneous, using Euler's identity. This implies that $\omega_1\theta_1, \omega_1\theta_2, d\theta_E, \eta_0$ are also $\mathcal{F}(S)$-independent.

Second, we claim that $\eta_0 \in (S \otimes V^* \otimes V)^G$, i.e., that η_0 has polynomial coefficients and is G-invariant. Since f and Q are G-invariant (as $e = 1$), so is η_0. Observe that

$$\begin{aligned} Q\eta_0 = &\Big(\tfrac{\partial Q}{\partial x_1}\tfrac{\partial Q}{\partial x_2} + f^{\deg Q - 2}x_1x_2\Big) \otimes x_1 \otimes v_1 + \Big(\big(\tfrac{\partial Q}{\partial x_2}\big)^2 + f^{\deg Q - 2}x_1^2\Big) \otimes x_2 \otimes v_1 \\ &+ \Big(\big(\tfrac{\partial Q}{\partial x_1}\big)^2 + f^{\deg Q - 2}x_2^2\Big) \otimes x_1 \otimes v_2 + \Big(\tfrac{\partial Q}{\partial x_1}\tfrac{\partial Q}{\partial x_2} + f^{\deg Q - 2}x_1x_2\Big) \otimes x_2 \otimes v_2\,. \end{aligned}$$

Notice that x_2 divides $\frac{\partial Q}{\partial x_1}$ (apply [13, Lemma 4] to ω_2), which implies that x_2 divides the first, third, and last coefficients in this expression. Then, as the factors of Q are relatively prime, x_2 does not divide $\frac{\partial Q}{\partial x_2}$, and we may rescale Q without loss of generality so that the term $x_1^{2\deg Q - 2}$ in $\big(\frac{\partial Q}{\partial x_2}\big)^2$ cancels with the same term in $f^{\deg Q - 2}x_1^2$, which implies that x_2 divides the second coefficient as well. Hence, $Q\eta_0 \in x_2 S \otimes V^* \otimes V$. Then as the reflecting hyperplanes of G are all in the same orbit by Corollary 4.2 and $Q\eta_0$ is invariant, $Q\eta_0 \in \ell_H S \otimes V^* \otimes V$ for any H in $\mathcal{A}$, and thus $Q\eta_0 \in QS \otimes V^* \otimes V$. Hence, η_0 indeed lies in $(S \otimes V^* \otimes V)^G$.

Finally, note that the polynomial degrees of $\omega_1\theta_1, \omega_1\theta_2, d\theta_E, \eta_0$ add to $2\deg Q$, and thus these differential derivations are an S^G-basis for $k=1$.

For $k=2$, the forms $\omega_1 d\theta_E$ and $\omega_1\eta_0$ are an S^G-basis since they are $\mathcal{F}(S)$-independent with polynomial degrees that add to $\deg Q$.

Hence $\theta_1, \theta_2, \omega_1\theta_1, \omega_1\theta_2, d\theta_E, \eta_0, \omega_1 d\theta_E, \omega_1\eta_0$ are an S^G-basis of $(S\otimes \wedge V^* \otimes V)^G$. Thus

$$(S\otimes \wedge V^*\otimes V)^G = R\text{-span}\{\theta_1, \theta_2, d\theta_E, \eta_0\}$$

for $R = S^G\text{-span}\{1\otimes 1, \omega_1\} \subset (S\otimes\wedge V^*)^G$ and the R-module $(S\otimes\wedge V^*\otimes V)^G$ has rank 4.

Alternatively, we note that $\theta_1, \theta_2, \omega_1\theta_1, \omega_1\theta_2, \eta_0, d\theta_E, (\omega_1 \curlywedge \omega_2)\theta_1, (\omega_1\curlywedge\omega_2)\theta_2$ also are an S^G-basis of $(S\otimes\wedge V^*\otimes V)^G$. This is because $(\omega_1\curlywedge\omega_2)\theta_1, (\omega_1\curlywedge\omega_2)\theta_2$ are an S^G-basis for $k=2$: they are $\mathcal{F}(S)$-independent with polynomial degrees that add to $\deg Q$. We compare with Theorem 6.2 and observe that this alternate S^G-basis of $(S\otimes\wedge V^*\otimes V)^G$ is

$$\{d\theta_E, \eta_0\} \cup \{\omega_I^{\curlywedge}\theta_1, \omega_I^{\curlywedge}\theta_2 : I\subset[2]\} \setminus \{\omega_2\theta_1, \omega_2\theta_2\}\,. \qquad \square$$

Theorem 7.1 and Corollary 6.4 imply the following.

Corollary 7.2. *Let $G\subset GL(V)$ be a reflection group with transvection root spaces all maximal and $\operatorname{char}\mathbb{F}$ arbitrary. If $(S\otimes V)^G$ is a free S^G-module, then so is $(S\otimes\wedge V^*\otimes V)^G$.*

Example 7.3. Let $G=\mathrm{SL}_2(\mathbb{F}_2)$. Here, $Q = x_1^2x_2 + x_1x_2^2$ and each G_H consists of exactly one transvection and the identity, so $e=b=\delta=1$. Then $(S\otimes V)^G$ is free over S^G with basis θ_1,θ_2 and $(S\otimes V^*)^G$ is free over S^G with basis ω_1,ω_2 (see Theorem 3.2) for

$$\theta_1 = x_1^2\otimes v_1 + x_2^2\otimes v_2, \quad \theta_2 = x_1\otimes v_1 + x_2\otimes v_2,$$
$$\omega_1 = x_2\otimes x_1 + x_1\otimes x_2, \quad \omega_2 = x_2^2\otimes x_1 + x_1^2\otimes x_2.$$

As $x_1^2+x_1x_2+x_2^2$ lies in S^G, the proof of Theorem 7.1 gives an S^G-basis of $(S\otimes\wedge V^*\otimes V)^G$:

$$\begin{aligned}
\theta_1 &= x_1^2\otimes 1\otimes v_1 + x_2^2\otimes 1\otimes v_2, \quad \theta_2 = x_1\otimes 1\otimes v_1 + x_2\otimes 1\otimes v_2,\\
\omega_1\theta_1 &= x_1^2x_2\otimes x_1\otimes v_1 + x_2^3\otimes x_1\otimes v_2 + x_1^3\otimes x_2\otimes v_1 + x_1x_2^2\otimes x_2\otimes v_2,\\
\omega_1\theta_2 &= x_1x_2\otimes x_1\otimes v_1 + x_2^2\otimes x_1\otimes v_2 + x_1^2\otimes x_2\otimes v_1 + x_1x_2\otimes x_2\otimes v_2,\\
d\theta_E &= 1\otimes x_1\otimes v_1 + 1\otimes x_2\otimes v_2,\\
\eta_0 &= (x_1+x_2)\otimes x_1\otimes v_1 + x_2\otimes x_1\otimes v_2 + x_1\otimes x_2\otimes v_1 + (x_1+x_2)\otimes x_2\otimes v_2,\\
\omega_1 d\theta_E &= x_1\otimes x_1\wedge x_2\otimes v_1 + x_2\otimes x_1\wedge x_2\otimes v_2\,, \quad \omega_1\eta_0 = x_1^2\otimes x_1\wedge x_2\otimes v_1 + x_2^2\otimes x_1\wedge x_2\otimes v_2\,.
\end{aligned}$$

8. Prime Fields

We now consider finite groups G acting on vector spaces over a prime field $\mathbb{F}=\mathbb{F}_p$ for a fixed prime p. We observe that $(S\otimes V)^G$, $(S\otimes\wedge V^*)^G$ and $(S\otimes\wedge V^*\otimes V)^G$ are free S^G-modules when G is a reflection group with transvection root spaces maximal and produce bases. Examples include $\mathrm{SL}_n(\mathbb{F}_p)$ and $\mathrm{GL}_n(\mathbb{F}_p)$, see the next section.

Reflection arrangements. We first examine arrangements over prime fields when the transvection roots spaces are maximal. Recall that $\mathcal{A}=\mathcal{A}(G)$ is the collection of reflecting hyperplanes of a group G acting linearly.

Lemma 8.1. *Let $G\subset GL(V)$ be a finite group acting on $V=\mathbb{F}_p^n$, for p a prime. If $H, H'\in\mathcal{A}$ with the transvection root space of H maximal, then $\mathcal{A}$ contains all hyperplanes in V containing $H\cap H'$.*

Proof. Say $H = \ker \ell_H$ and $H' = \ker \ell_{H'}$ for $\ell_H, \ell_{H'} \in V^*$. A hyperplane in V containing $H \cap H'$ must be the kernel of $\ell_{H'} + c\,\ell_H$ for some $c \in \mathbb{F}_p$. As the transvection root space of H is maximal, G contains a transvection t about H whose root vector v_t lies outside of H', i.e., $\ell_{H'}(v_t) \neq 0$. Set $a = (\ell_{H'}(v_t))^{-1}c$ in $\mathbb{F}_p$ and regard a as an integer. A straightforward calculation confirms that the kernel of $\ell_{H'} + c\ell_H$ is the reflecting hyperplane of $t^{-a}\, s'\, t^a$ for any reflection s' in G about H' and thus lies in $\mathcal{A}$. □

Proposition 8.2. *Let $G \subset GL(V)$ be a finite group acting on $V = \mathbb{F}_p^n$, for p a prime, with transvection root spaces all maximal. Then the reflection arrangement of G coincides with that for the general linear group (embedded in $GL(V)$) of some subspace W of V:*

$$\mathcal{A}(G) = \mathcal{A}(GL(W)) \qquad \textit{for some } GL(W) \subset GL(V)\,.$$

Thus there is a basis $x_1, \ldots, x_n$ of V^* with $\mathcal{A}(G)$ defined by, for some m,

$$Q = x_1\Big(\prod_{\alpha_1 \in \mathbb{F}_p} x_2 + \alpha_1 x_1\Big)\Big(\prod_{\alpha_1,\alpha_2 \in \mathbb{F}_p} x_3 + \alpha_2 x_2 + \alpha_1 x_1\Big)\cdots\Big(\prod_{\alpha_1,\ldots,\alpha_{m-1} \in \mathbb{F}_p} x_m + \alpha_{m-1}x_{m-1} + \cdots + \alpha_1 x_1\Big)\,.$$

Proof. Let $H_1 \in \mathcal{A} = \mathcal{A}(G)$ be arbitrary and set $\mathcal{A}_1 = \{H_1\}$. Inductively choose some $H_i \in \mathcal{A}\backslash\mathcal{A}_{i-1}$ and set $\mathcal{A}_i = \{H \in \mathcal{A} : H \supset H_1 \cap \cdots \cap H_i\}$ to obtain a maximum set of hyperplanes $H_1, \ldots, H_m$ for which $\mathcal{A}_m = \mathcal{A}$. Choose x_i in V^* so that $H_i = \ker(x_i) \in \mathcal{A}$ and notice that $x_1, \ldots, x_m$ is $\mathbb{F}$-independent since $\dim(H_1 \cap \ldots \cap H_i) = n - i$ for all $i \leq m$. We extend to a basis $x_1, \ldots, x_n$ of V^*. By Lemma 8.1, any nonzero linear combination of linear forms defining hyperplanes in $\mathcal{A}$ defines a hyperplane again in $\mathcal{A}$. Thus for each $i \leq m$, $\mathcal{A}_i = \{H : \ell_H \in \mathbb{F}_p\text{-span}\{x_1, \ldots, x_i\}\}$, and the claim follows. □

Free arrangements. Recall that an arrangement of hyperplanes $\mathcal{A}$ is *free* if the set of derivations $D(\mathcal{A})$ along the arrangement is a free S-module, see [15], where

$$D(\mathcal{A}) = \{\theta \in \mathrm{Der}_S : \theta(\ell_H) \in \ell_H S \text{ for all } H \in \mathcal{A}\}\,.$$

(Recall that we identify $\sum_i f_i \otimes v_i$ in $S \otimes V$ with the derivation $\sum_i f_i \otimes \partial/\partial x_i$.) Bases for the free modules in the next corollary are given in Proposition 5.3 and Theorem 6.2 using the derivations in the proof. Also see the proofs of Theorem 7.1 and Proposition 3.8.

Corollary 8.3. *If $G \subset GL(V)$ is a finite group acting on $V = \mathbb{F}_p^n$ for p a prime with transvection root spaces all maximal, then $\mathcal{A}(G)$ is a free arrangement. If, in addition, G is a reflection group, then $(S \otimes V)^G$, $(S \otimes \wedge V^*)^G$, and $(S \otimes \wedge V^* \otimes V)^G$ are free S^G-modules.*

Proof. By Proposition 8.2, $\mathcal{A} = \mathcal{A}(G) = \mathcal{A}(\mathrm{GL}(W))$ for a subspace W of V of dimension m. We use the basis $x_1, \ldots, x_n$ of V^* of Proposition 8.2 and dual basis $v_1, \ldots, v_n$ of V and set

$$\theta_i = \begin{cases} \displaystyle\sum_{j=1}^{n} x_j^{p^{m-i}} \otimes v_j & \text{for } 1 \leq i \leq m\,, \\ 1 \otimes v_i & \text{for } m < i \leq n \end{cases}$$

so that $\det \mathrm{Coef}(\theta_1, \ldots, \theta_n) \doteq Q$. Then as each θ_i lies in $D(\mathcal{A})$, the θ_i generate $D(\mathcal{A})$ as an S-module and $\mathcal{A}$ is a free arrangement by the original Saito's Criterion [15, Theorem 4.19]. Now assume further that G is a reflection group. Notice that each θ_i for $i \leq m$ is invariant under $\mathrm{GL}_n(\mathbb{F}_p)$ (see Section 9) and that θ_i for $m < i$ is invariant under each reflection of G since $v_{m+1}, \ldots, v_n$ lie in $\bigcap_{H \in \mathcal{A}} H$. Hence $\theta_1, \ldots, \theta_n$ are G-invariant and are an S^G-basis of $(S \otimes V)^G$ by Theorem 3.2. Then $(S \otimes \wedge V^*)^G$ and $(S \otimes \wedge V^* \otimes V)^G$ are both free S^G-modules by Corollary 5.6 and Corollary 7.2. □

9. Special and General Linear Groups and Groups in between

We now turn our attention to the special linear group, the general linear group, and all groups in between over a finite field $\mathbb{F}_q$ for q a prime power. Let G be a group with $\mathrm{SL}_n(\mathbb{F}_q) \subset G \subset \mathrm{GL}_n(\mathbb{F}_q)$. Then G is generated by reflections, and, as G contains $\mathrm{SL}_n(\mathbb{F}_q)$, each transvection root space for G is maximal and there is a single orbit of reflecting hyperplanes (see Corollary 4.2). The maximal order of a diagonalizable reflection in G is $e := |G : \mathrm{SL}_n(\mathbb{F}_q)|$. Here, $\mathcal{A} = \mathcal{A}(G)$ is the collection of all hyperplanes H in $V = \mathbb{F}_q^n$ and its defining polynomial $Q = \prod_{H\in\mathcal{A}} \ell_H$ thus has degree $|\mathcal{A}| = [n]_q = 1 + q + \cdots + q^{n-1}$.

Invariant polynomials. Basic invariant polynomials $f_1, \ldots, f_n$ with $S^G = \mathbb{F}[f_1, \ldots, f_n]$ are given in terms of the classical Dickson invariants $D_{n,i}$ (see [24] and [21]) with $\deg D_{n,i} = q^n - q^i$ for $i = 0, \ldots, n-1$:

$$f_1 = Q^e \quad \text{and} \quad f_i = D_{n,i-1} \quad \text{for } 2 \le i \le n\,.$$

Invariant derivations. Here, $(S \otimes V)^G$ is a free S^G-module with basis

$$\theta_i = \sum_{j=1}^{n} x_j^{q^{n-i}} \otimes v_j \quad \text{for } 1 \le i \le n$$

with respect to a fixed ordered basis $v_1, \ldots, v_n$ of V and dual basis $x_1, \ldots, x_n$ of V^* (see [15, Example 4.24]) since $Q = \det \mathrm{Coef}(\theta_1, \ldots, \theta_n)$ after rescaling Q if necessary (see Theorem 3.2).

Invariant 1-forms. Proposition 5.3 gives a dual S^G-basis of $(S \otimes V^*)^G$: explicitly, let $\omega_1, \ldots, \omega_n$ in $S \otimes V^*$ be the 1-forms whose coefficient matrix is (for t indicating transpose)

$$\mathrm{Coef}(\omega_1, \ldots, \omega_n) = Q^e \left(\mathrm{Coef}(\theta_1, \ldots, \theta_n)\right)^{-t}.$$

Then $\det \mathrm{Coef}(\omega_1, \ldots, \omega_n) = Q^{en-1}$ and $\omega_1, \ldots, \omega_n$ are a free S^G-basis of $(S \otimes V^*)^G$ by Theorem 3.2. These moreover generate $(S \otimes \wedge V^*)^G$ via the twisted wedging of Eq. (4.4): $(S \otimes \wedge V^*)^G = \curlywedge_{S^G}\{\omega_1, \ldots, \omega_n\}$ (see Theorem 4.6 and [13]). See [13, Section 6.2] for basic 1-forms in terms of the exterior derivatives df_i of the Dickson invariants.

Numerology. For $m_i = \deg \omega_i = e[n]_q - q^{n-i}$ and $m_i^* = \deg \theta_i = q^{n-i}$ (see Corollary 5.6),

$$m_i + m_i^* = e|\mathcal{A}|\,.$$

Explicitly, the duality gives (also see Remark 4.7 and Remark 5.7)

$$\begin{aligned} m_i &= [n]_q - q^{n-i} &\quad&\text{and}\quad & m_i^* &= q^{n-i} &\quad& \text{for } G = \mathrm{SL}_n(\mathbb{F}_q)\,, \text{ and}\\ m_i &= (q-1)[n]_q - q^{n-i} &&\text{and}& m_i^* &= q^{n-i} && \text{for } G = \mathrm{GL}_n(\mathbb{F}_q)\,. \end{aligned}$$

Invariant differential derivations. For $G = \mathrm{SL}_2(\mathbb{F}_2)$, we construct in Example 7.3 an explicit basis for $(S \otimes \wedge V^* \otimes V)^G$ as a free S^G-module. Theorem 6.2 and Theorem 7.1 imply a similar result for all other groups between $\mathrm{SL}_n(\mathbb{F}_q)$ and $\mathrm{GL}_n(\mathbb{F}_q)$:

Corollary 9.1. *Let G be a group with $SL_n(\mathbb{F}_q) \subset G \subset GL_n(\mathbb{F}_q)$ and $G \ne SL_2(\mathbb{F}_2)$. Then $(S \otimes \wedge V^* \otimes V)^G$ is a free S^G-module with basis*

$$\{d\theta_E\} \cup \big\{(\omega_{i_1} \curlywedge \cdots \curlywedge \omega_{i_k})\,\theta_j : 1 \le i_1 < \ldots < i_k \le n, 1 \le j \le n, 0 \le k \le n\big\} \setminus \{\omega_n\theta_n\}\,.$$

Example 9.2. For the reflection group $G = \mathrm{SL}_2(\mathbb{F}_3)$ acting on $V = \mathbb{F}_3^2$,

$$\begin{array}{lll} \text{basic derivations} & \theta_1 = x_1^3 \otimes v_1 + x_2^3 \otimes v_2\,, & \theta_2 = x_1 \otimes v_1 + x_2 \otimes v_2 \ \text{ and} \\ \text{basic 1-forms} & \omega_1 = x_2 \otimes x_1 - x_1 \otimes x_2\,, & \omega_2 = -x_2^3 \otimes x_1 + x_1^3 \otimes x_2 \end{array}$$

generate $(S \otimes V)^G$ and $(S \otimes V^*)^G$, respectively, as free S^G-modules. Then the S^G-module $(S \otimes \wedge V^* \otimes V)^G$ is also free with basis

$$\begin{aligned}
\theta_1 &= x_1^3 \otimes 1 \otimes v_1 + x_2^3 \otimes 1 \otimes v_2\,, \quad \theta_2 = x_1 \otimes 1 \otimes v_1 + x_2 \otimes 1 \otimes v_2\,,\\
\omega_1\theta_1 &= x_1^3 x_2 \otimes x_1 \otimes v_1 + x_2^4 \otimes x_1 \otimes v_2 - x_1^4 \otimes x_2 \otimes v_1 - x_1 x_2^3 \otimes x_2 \otimes v_2\,,\\
\omega_1\theta_2 &= x_1 x_2 \otimes x_1 \otimes v_1 + x_2^2 \otimes x_1 \otimes v_2 - x_1^2 \otimes x_2 \otimes v_1 - x_1 x_2 \otimes x_2 \otimes v_2\,,\\
\omega_2\theta_1 &= -x_1^3 x_2^3 \otimes x_1 \otimes v_1 - x_2^6 \otimes x_1 \otimes v_2 + x_1^6 \otimes x_2 \otimes v_1 + x_1^3 x_2^3 \otimes x_2 \otimes v_2\,,\\
d\theta_E &= 1 \otimes x_1 \otimes v_1 + 1 \otimes x_2 \otimes v_2\,,\\
(\omega_1 \curlywedge \omega_2)\theta_1 &= x_1^3 \otimes x_1 \wedge x_2 \otimes v_1 + x_2^3 \otimes x_1 \wedge x_2 \otimes v_2\,,\\
(\omega_1 \curlywedge \omega_2)\theta_2 &= x_1 \otimes x_1 \wedge x_2 \otimes v_1 + x_2 \otimes x_1 \wedge x_2 \otimes v_2\,.
\end{aligned}$$

Here, $S^G = \mathbb{F}_3[f_1, f_2]$ for $f_1 = x_1^3 x_2 - x_1 x_2^3$ and $f_2 = x_1^6 + x_1^4 x_2^2 + x_1^2 x_2^4 + x_2^6$.

Example 9.3. For the reflection group $G = \mathrm{GL}_2(\mathbb{F}_3)$ acting on $V = \mathbb{F}_3^2$, the S^G-modules $(S \otimes V)^G$ and $(S \otimes V^*)^G$ are both free with respective bases θ_1, θ_2 (basic derivations) and ω_1, ω_2 (basic 1-forms) given by

$$\begin{aligned}
&\theta_1 = x_1^3 \otimes v_1 + x_2^3 \otimes v_2\,, \quad \theta_2 = x_1 \otimes v_1 + x_2 \otimes v_2\,,\\
&\omega_1 = (x_1^3 x_2^2 - x_1 x_2^4) \otimes x_1 + (x_1^2 x_2^3 - x_1^4 x_2) \otimes x_2,\ \omega_2 = (x_1 x_2^6 - x_1^3 x_2^4) \otimes x_1 + (x_1^6 x_2 - x_1^4 x_2^3) \otimes x_2\,.
\end{aligned}$$

The S^G-module $(S \otimes \wedge V^* \otimes V)^G$ is then free with basis

$$\begin{aligned}
\theta_1 &= x_1^3 \otimes 1 \otimes v_1 + x_2^3 \otimes 1 \otimes v_2\,, \quad \theta_2 = x_1 \otimes 1 \otimes v_1 + x_2 \otimes 1 \otimes v_2\,,\\
\omega_1\theta_1 &= (x_1^6 x_2^2 - x_1^4 x_2^4) \otimes x_1 \otimes v_1 + (x_1^3 x_2^5 - x_1 x_2^7) \otimes x_1 \otimes v_2\\
&\quad + (x_1^5 x_2^3 - x_1^7 x_2) \otimes x_2 \otimes v_1 + (x_1^2 x_2^6 - x_1^4 x_2^4) \otimes x_2 \otimes v_2\,,\\
\omega_1\theta_2 &= (x_1^4 x_2^2 - x_1^2 x_2^4) \otimes x_1 \otimes v_1 + (x_1^3 x_2^3 - x_1 x_2^5) \otimes x_1 \otimes v_2\\
&\quad + (x_1^3 x_2^3 - x_1^5 x_2) \otimes x_2 \otimes v_1 + (x_1^2 x_2^4 - x_1^4 x_2^2) \otimes x_2 \otimes v_2\,,\\
\omega_2\theta_1 &= (x_1^4 x_2^6 - x_1^6 x_2^4) \otimes x_1 \otimes v_1 + (x_1 x_2^9 - x_1^3 x_2^7) \otimes x_1 \otimes v_2\\
&\quad + (x_1^9 x_2 - x_1^7 x_2^3) \otimes x_2 \otimes v_1 + (x_1^6 x_2^4 - x_1^4 x_2^6) \otimes x_2 \otimes v_2,\\
d\theta_E &= 1 \otimes x_1 \otimes v_1 + 1 \otimes x_2 \otimes v_2\,,\\
(\omega_1 \curlywedge \omega_2)\theta_1 &= (x_1^6 x_2 - x_1^4 x_2^3) \otimes x_1 \wedge x_2 \otimes v_1 + (x_1^3 x_2^4 - x_1 x_2^6) \otimes x_1 \wedge x_2 \otimes v_2\,,\\
(\omega_1 \curlywedge \omega_2)\theta_2 &= (x_1^4 x_2 - x_1^2 x_2^3) \otimes x_1 \wedge x_2 \otimes v_1 + (x_1^3 x_2^2 - x_1 x_2^4) \otimes x_1 \wedge x_2 \otimes v_2\,.
\end{aligned}$$

Here, $S^G = \mathbb{F}_3[f_1, f_2]$ for $f_1 = x_1^6 + x_1^4 x_2^2 + x_1^2 x_2^4 + x_2^6$ and $f_2 = x_1^6 x_2^2 + x_1^4 x_2^4 + x_1^2 x_2^6$.

Appendix A.

The technical analysis in this appendix provides the heavy lifting for determining the Saito criterion for invariant differential derivations in Section 3. Throughout this section, we fix a nontrivial finite group $G \subset \mathrm{GL}(V)$ acting on $V = \mathbb{F}^n$ that fixes a single hyperplane $H = \ker \ell$ in V for some linear form ℓ in V^*. We fix $e = e_H \geq 1$ and $b = b_H \geq 0$ throughout and use the basis $v_1, \ldots, v_n$ of V with dual basis $x_1, \ldots, x_n$ of V^* as in Eq. (2.2), as well as the (possibly empty) set of transvections $t_1, \ldots, t_b$ in G and an element s in G with either $s = 1_G$ when $e = 1$ or s is a diagonalizable reflection of maximal order $e > 1$ in G.

Action on basis elements. We record the action of s and each transvection t_m on basis elements v_i of V and x_I of $\wedge V^*$. For a fixed m and $I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$ with $i_1 < \ldots < i_k$, $n \notin I$, and $m \in I$, define $\varepsilon_{I,m} = \pm 1$ by

$$x_{\sigma(I)} = \varepsilon_{I,m}\ x_{\sigma(i_1)} \wedge \cdots \wedge x_{\sigma(i_k)} \qquad \text{for the transposition } \sigma = (m\ n)\,.$$

Also, set $\lambda = \det(s)$. Then for $1 \le m \le b$,

$$t_m(x_I) = \begin{cases} x_I & \text{when } n \in I \text{ or } m \notin I, \\ x_I - \varepsilon_{I,m} x_{\sigma(I)} & \text{when } n \notin I \text{ and } m \in I, \end{cases} \qquad t_m(v_j) = \begin{cases} v_j & \text{when } j \neq n\,, \\ v_m + v_n & \text{when } j = n\,, \end{cases}$$

$$s(x_I) = \begin{cases} x_I & \text{when } n \notin I, \\ \lambda^{-1} x_I & \text{when } n \in I, \end{cases} \qquad \text{and} \quad s(v_j) = \begin{cases} v_j & \text{when } j \neq n\,, \\ \lambda v_n & \text{when } j = n\,. \end{cases}$$

Note that $\varepsilon_{I,m} = 1$ and $t_m(x_I) = x_m - x_n$ when $I = \{m\}$.

Action on polynomials. We require some straightforward observations.

Lemma A.1. *For any reflection g about $H = \ker \ell$ and any polynomial f in S, ℓ divides $g(f) - f$. Also, ℓ^2 divides $g(f) - f$ whenever ℓ divides f.*

Lemma A.2. *Let $\det(s) = \lambda$ of order $e \ge 1$. Then for any polynomial f,*

(a) $s(f) = \lambda f$ implies ℓ^{e-1} divides f,
(b) $s(f) = f$ and ℓ divides f implies ℓ^e divides f,
(c) $s(f) = \lambda^{-1} f$ and $\lambda \neq 1$ implies ℓ divides f, and
(d) $s(f) = \lambda^{-1} f$ and ℓ^2 divides f implies ℓ^{e+1} divides f.

Proof. We prove part (a); the rest follows from similar arguments. Since $s(x_n) = \lambda^{-1} x_n$ and s fixes $x_1, \ldots, x_{n-1}$ as well as x_n^e, the degree in x_n of each monomial appearing in f must be $-1 \mod e$, and thus $\ell^{e-1} = x_n^{e-1}$ divides each monomial. □

Lemma A.3. *Say $1 \le m \le b$. If a monomial $M = x_1^{a_1} \cdots x_{n-1}^{a_{n-1}} x_n$ appears in $t_m(f) - f$ with nonzero coefficient c, then $a_m + 1 \neq 0$ in $\mathbb{F}$ and $M x_m / x_n$ appears in f with nonzero coefficient $-(a_m + 1)^{-1} c$.*

Proof. For any monomial $x_1^{a_1} \cdots x_n^{a_n}$,

$$t_m(x_1^{a_1} \cdots x_n^{a_n}) - x_1^{a_1} \cdots x_n^{a_n} = -a_m x_1^{a_1} \cdots x_{m-1}^{a_{m-1}} x_m^{a_m - 1} x_{m+1}^{a_{m+1}} \cdots x_n^{a_n + 1} \ + \text{ other terms.}$$

The claim then follows from the observation that this expression is zero for $a_m = 0$, and otherwise all monomials appearing in this expression have degree in x_n strictly greater than a_n, degree in x_m strictly less than a_m, and unchanged degrees in the other variables. □

Main lemma. At last, the following lemma analyzes the polynomial coefficients of an invariant differential derivation. Recall that $[b] = \{1, \ldots, b\}$.

Lemma A.4. *For any differential derivation $\eta = \sum_{I \in \binom{[n]}{k},\, 1 \le j \le n} f_{I,j} \otimes x_I \otimes v_j$ in $(S \otimes \wedge^k V^* \otimes V)^G$,*

a) ℓ divides $f_{I,n}$ when $n \notin I$,
b) ℓ^e divides $f_{I,n}$ when $n \in I$ and $I \cap [b] \neq \varnothing$,
c) ℓ^{e-1} divides $f_{I,j}$ for $j < n$ when $n \in I$,
d) ℓ^e divides $f_{I,j}$ for $j < n$ when $n \notin I$ and $I \cap [b] \neq \varnothing$ and $I \cap [b] \neq \{j\}$,
e) ℓ^e divides $f_{I,j} - \varepsilon_{I,m} f_{\sigma(I),n}$ for $j = m \le b$ when $n \notin I$ and $I \cap [b] = \{m\}$, and
f) ℓ^{e+1} divides $f_{I,n}$ when $n \notin I$ and $I \cap [b] \neq \varnothing$, unless G consists of exactly one transvection and the identity element.

Proof. We take all sums over subsets $I \in \binom{[n]}{k}$ and $1 \le j \le n$ as indicated.

Action of transvections. Consider the transvection t_m for $m \in [b]$ when $b > 0$ and set $\sigma = (m\ n)$ and $\varepsilon_I = \varepsilon_{I,m}$:

$$\begin{aligned} t_m(\eta) &= \sum_{I,j} t_m(f_{I,j}) \otimes t_m(x_I) \otimes t_m(v_j) \\ &= \sum_{\substack{I,j:\\ n\in I \text{ or } m\notin I,\\ j\neq n}} t_m(f_{I,j}) \otimes x_I \otimes v_j \ + \sum_{\substack{I:\\ n\in I \text{ or } m\notin I}} t_m(f_{I,n}) \otimes x_I \otimes (v_m + v_n) \\ &\quad + \sum_{\substack{I,j:\\ n\notin I,\, m\in I,\\ j\neq n}} t_m(f_{I,j}) \otimes (x_I - \varepsilon_I x_{\sigma(I)}) \otimes v_j \ + \sum_{\substack{I:\\ n\notin I,\, m\in I}} t_m(f_{I,n}) \otimes (x_I - \varepsilon_I x_{\sigma(I)}) \otimes (v_m + v_n)\,. \end{aligned}$$

We reindex and regroup to express $t_m(\eta)$ as

$$\begin{aligned} &\sum_{\substack{I,j:\\ n\notin I \text{ or } m\in I,\\ j\neq m}} t_m(f_{I,j}) \otimes x_I \otimes v_j \ + \sum_{\substack{I:\\ n\notin I \text{ or } m\in I}} \big(t_m(f_{I,m}) + t_m(f_{I,n})\big) \otimes x_I \otimes v_m \\ &\quad + \sum_{\substack{I,j:\\ n\in I,\, m\notin I,\\ j\neq m}} \big(t_m(f_{I,j}) - \varepsilon_{\sigma(I)} t_m(f_{\sigma(I),j})\big) \otimes x_I \otimes v_j \\ &\quad + \sum_{\substack{I:\\ n\in I,\, m\notin I}} \big(t_m(f_{I,m}) + t_m(f_{I,n}) - \varepsilon_{\sigma(I)} t_m(f_{\sigma(I),m}) - \varepsilon_{\sigma(I)} t_m(f_{\sigma(I),n})\big) \otimes x_I \otimes v_m\,. \end{aligned}$$

We equate the polynomial coefficients of η and $t_m(\eta)$ and deduce that

$$f_{I,j} = \begin{cases} t_m(f_{I,j}) & \text{for } j \neq m \text{ when } n \notin I \quad \text{or } m \in I, \\ t_m(f_{I,m}) + t_m(f_{I,n}) & \text{for } j = m \text{ when } n \notin I \quad \text{or } m \in I, \\ t_m(f_{I,j}) - \varepsilon_{\sigma(I)} t_m(f_{\sigma(I),j}) & \text{for } j \neq m \text{ when } n \in I \text{ and } m \notin I, \\ t_m(f_{I,m}) + t_m(f_{I,n}) - \varepsilon_{\sigma(I)} t_m(f_{\sigma(I),m}) - \varepsilon_{\sigma(I)} t_m(f_{\sigma(I),n}) & \text{for } j = m \text{ when } n \in I \text{ and } m \notin I. \end{cases}$$

We solve for $t_m(f_{I,j})$ one case at a time and conclude that

$$t_m(f_{I,j}) = \begin{cases} f_{I,j} & \text{for } j \neq m \text{ when } n \notin I \quad \text{or } m \in I, \\ f_{I,m} - f_{I,n} & \text{for } j = m \text{ when } n \notin I \quad \text{or } m \in I, \\ f_{I,j} + \varepsilon_{\sigma(I)} f_{\sigma(I),j} & \text{for } j \neq m \text{ when } n \in I \text{ and } m \notin I, \\ f_{I,m} - f_{I,n} + \varepsilon_{\sigma(I)} f_{\sigma(I),m} - \varepsilon_{\sigma(I)} f_{\sigma(I),n} & \text{for } j = m \text{ when } n \in I \text{ and } m \notin I. \end{cases} \tag{A.5}$$

Action of diagonalizable reflection. Since s is diagonal with $\det(s) = \lambda$ of order $e \ge 1$,

$$\begin{aligned} s(\eta) = \sum_{I,j} s(f_{I,j}) \otimes s(x_I) \otimes s(v_j) &= \sum_{\substack{I,j:n\notin I\\ j\neq n}} s(f_{I,j}) \otimes x_I \otimes v_j \ + \sum_{I:n\notin I} \lambda s(f_{I,n}) \otimes x_I \otimes v_n \\ &\quad + \sum_{\substack{I,j:n\in I\\ j\neq n}} \lambda^{-1} s(f_{I,j}) \otimes x_I \otimes v_j \ + \sum_{I:n\in I} s(f_{I,n}) \otimes x_I \otimes v_n. \end{aligned}$$

We equate the polynomial coefficients of η and $s(\eta)$ to see that

$$s(f_{I,j}) = \begin{cases} f_{I,j} & \text{for } j \neq n \text{ when } n \notin I, \\ \lambda^{-1} f_{I,n} & \text{for } j = n \text{ when } n \notin I, \\ \lambda f_{I,j} & \text{for } j \neq n \text{ when } n \in I, \\ f_{I,n} & \text{for } j = n \text{ when } n \in I. \end{cases} \tag{A.6}$$

Now we use equations Eq. (A.5) and Eq. (A.6) to show ℓ to certain powers divides various $f_{I,j}$ using the fact that G contains either a diagonalizable reflection or a transvection.

Parts a) through e). For a), fix I with $n \notin I$. As G is nontrivial, either G contains a transvection t_m or $s \neq 1_G$. If G contains t_m, then $f_{I,n} = f_{I,m} - t_m(f_{I,m})$ by Eq. (A.5) so is divisible by ℓ by Lemma A.1. If $s \neq 1_G$, then $s(f_{I,n}) = \lambda^{-1} f_{I,n}$ for $\lambda \neq 1$ by Eq. (A.6), so ℓ divides $f_{I,n}$ by Lemma A.2(c). Either way, ℓ divides $f_{I,n}$. The proof of parts b), c), and d) are similar. For part e), fix $j = m \leq b$ and I with $n \notin I$ and $I \cap [b] = \{m\}$. Then for $\sigma = (m\ n)$ and $\varepsilon_I = \varepsilon_{I,m}$,

$$t_m(f_{\sigma(I),j}) = f_{\sigma(I),j} - f_{\sigma(I),n} + \varepsilon_I f_{I,j} - \varepsilon_I f_{I,n}$$

by Eq. (A.5). Since ℓ divides $t_m(f_{\sigma(I),j}) - f_{\sigma(I),j}$ by Lemma A.1 and also $f_{I,n}$ by part a), it must divide $-f_{\sigma(I),n} + \varepsilon_I f_{I,j}$ and hence also $f_{I,j} - \varepsilon_I f_{\sigma(I),n}$. Further, ℓ^e divides $f_{I,j} - \varepsilon_I f_{\sigma(I),n}$ by Lemma A.2(b) since it is fixed by s (see Eq. (A.6)) and part e) follows.

Part f). Complications arise when $\operatorname{char} \mathbb{F} = 2$. For part f), assume G does not consist of exactly one transvection and the identity, and fix I with $n \notin I$ and $I \cap [b] \neq \varnothing$. Then G contains a transvection t_m for some $m \in I \cap [b]$ (so $b \neq 0$) and

1) $\operatorname{char} \mathbb{F} \neq 2$, or
2) $e > 1$, or
3) $b > 1$, or
4) $\operatorname{char} \mathbb{F} = 2$, $e = 1$, and $b = 1$, but G contains multiple transvections.

In each case, we will show that ℓ^2 divides $f_{I,n}$. Then as $s(f_{I,n}) = \lambda^{-1} f_{I,n}$ by Eq. (A.6), Lemma A.2(d) will imply that ℓ^{e+1} divides $f_{I,n}$ and the claim for part f) will follow. We fix $m \in I \cap [b]$, $\sigma = (m\ n)$, and $\varepsilon_I = \varepsilon_{I,m}$.

Case 1: $\operatorname{char} \mathbb{F} \neq 2$. Suppose that ℓ^2 does not divide $f_{I,n}$. Notice ℓ divides $f_{I,n}$ by part a) so some monomial M of degree 1 in x_n appears in $f_{I,n}$ with nonzero coefficient $c \in \mathbb{F}$. As $t_m(f_{I,n}) = f_{I,n}$ by Eq. (A.5),

$$f_{I,n} \in S^{G'} = \mathbb{F}[x_m^p - x_m x_n^{p-1}, x_i : i \neq m] \quad \text{for } G' = \langle t_m \rangle$$

(e.g., see [21]), and the degree of M in x_m is a multiple of p (as x_n divides $f_{I,n}$). By Eq. (A.5),

$$t_m(f_{I,m}) = f_{I,m} - f_{I,n} \quad \text{and} \quad t_m(f_{\sigma(I),n}) = f_{\sigma(I),n} + \varepsilon_I f_{I,n}\,,$$

so $M x_m / x_n$ appears in $f_{I,m}$ and $f_{\sigma(I),n}$ with nonzero coefficients c and $-\varepsilon_I c$, respectively, by Lemma A.3. Thus $\ell = x_n$ does not divide $-f_{\sigma(I),n} + \varepsilon_I f_{I,m}$ (as $\operatorname{char} \mathbb{F} \neq 2$). But

$$t_m(f_{\sigma(I),m}) = f_{\sigma(I),m} - f_{\sigma(I),n} + \varepsilon_I f_{I,m} - \varepsilon_I f_{I,n}\,,$$

and ℓ divides $f_{I,n}$, so ℓ must divide $-f_{\sigma(I),n} + \varepsilon_I f_{I,m}$ by Lemma A.1 giving a contradiction. Thus ℓ^2 divides $f_{I,n}$.

Case 2: $\mathbf{e > 1}$. Here, $s(f_{\sigma(I),m}) = \lambda f_{\sigma(I),m}$ by Eq. (A.6) so ℓ divides $f_{\sigma(I),m}$ by Lemma A.2(a). Also, by Eq. (A.5),

$$t_m(f_{\sigma(I),m}) = f_{\sigma(I),m} - f_{\sigma(I),n} + \varepsilon_I f_{I,m} - \varepsilon_I f_{I,n},$$

so ℓ^2 divides $-f_{\sigma(I),n} + \varepsilon_I f_{I,m} - \varepsilon_I f_{I,n}$ by Lemma A.1. Recall again that ℓ divides $f_{I,n}$ by part a) so ℓ also divides $-f_{\sigma(I),n} + \varepsilon_I f_{I,m}$. Further, s fixes $-f_{\sigma(I),n} + \varepsilon_I f_{I,m}$ (see Eq. (A.6)) so ℓ^2 divides $-f_{\sigma(I),n} + \varepsilon_I f_{I,m}$ by Lemma A.2(b) as $e > 1$. Therefore ℓ^2 divides $f_{I,n}$. Finally, $s(f_{I,n}) = \lambda^{-1} f_{I,n}$, so ℓ^{e+1} divides $f_{I,n}$ by Lemma A.2(d).

Case 3: $\mathbf{b > 1}$. First, if $I \cap [b] \neq \{m\}$, then ℓ divides $f_{I,m}$ by part d). Then by Eq. (A.5), $f_{I,n} = f_{I,m} - t_m(f_{I,m})$, so ℓ^2 divides $f_{I,n}$ by Lemma A.1. Otherwise, if $I \cap [b] = \{m\}$, take $m' \in [b]$ with $m \neq m'$, so $m' \notin I$. Then by Eq. (A.5), $t_m(f_{\sigma(I),m'}) = f_{\sigma(I),m'} + \varepsilon_I f_{I,m'}$, so ℓ divides $f_{I,m'}$ and ℓ^2 divides $f_{I,n} = f_{I,m'} - t_{m'}(f_{I,m'})$ by Lemma A.1.

Case 4: char $\mathbb{F} = 2$, $\mathbf{e = 1}$, $\mathbf{b = 1}$, G contains multiple transvections. In this case, $m = 1$ and G contains the transvection t_1 as well as a transvection $t_1^{(\alpha)}$ with root vector αv_1 for some $\alpha \in \mathbb{F}$ that is not 0 or 1:

$$t_1 = \begin{pmatrix} 1 & & 1 \\ & \ddots & \\ & & 1 \end{pmatrix}, \quad t_1^{(\alpha)} = \begin{pmatrix} 1 & & \alpha \\ & \ddots & \\ & & 1 \end{pmatrix}.$$

Then since char $\mathbb{F} = 2$, for $\sigma = (1\ n)$,

$$t_1^{(\alpha)}(x_I) = \begin{cases} x_I & \text{when } n \in I \quad \text{or } 1 \notin I, \\ x_I + \alpha x_{\sigma(I)} & \text{when } n \notin I \text{ and } 1 \in I, \end{cases} \qquad \text{and} \qquad t_1^{(\alpha)}(v_j) \; = \begin{cases} v_j & \text{when } j \neq n, \\ \alpha v_1 + v_n & \text{when } j = n\,. \end{cases}$$

Taking sums over subsets $I \in \binom{[n]}{k}$ and $1 \leq j \leq n$, we observe after some computation that

$$\begin{aligned} t_1^{(\alpha)}(\eta) = & \sum_{\substack{I,j: \\ n \notin I \text{ or } 1 \in I, \\ j \neq 1}} t_1^{(\alpha)}(f_{I,j}) \otimes x_I \otimes v_j \;+ \sum_{\substack{I: \\ n \notin I \text{ or } 1 \in I}} \big(t_1^{(\alpha)}(f_{I,1}) + \alpha\, t_1^{(\alpha)}(f_{I,n})\big) \otimes x_I \otimes v_1 \\ & + \sum_{\substack{I,j: \\ n \in I, 1 \notin I, \\ j \neq 1}} \big(t_1^{(\alpha)}(f_{I,j}) + \alpha\, t_1^{(\alpha)}(f_{\sigma(I),j})\big) \otimes x_I \otimes v_j \\ & + \sum_{\substack{I: \\ n \in I, 1 \notin I}} \big(t_1^{(\alpha)}(f_{I,1}) + \alpha\, t_1^{(\alpha)}(f_{I,n}) + \alpha\, t_1^{(\alpha)}(f_{\sigma(I),1}) + \alpha^2\, t_1^{(\alpha)}(f_{\sigma(I),n})\big) \otimes x_I \otimes v_1\,. \end{aligned}$$

We equate polynomial coefficients of η and $t_1^{(\alpha)}(\eta)$ to deduce that

$$\text{(A.7)} \qquad t_1^{(\alpha)}(f_{I,j}) = \begin{cases} f_{I,j} & \text{for } j \neq 1 \text{ when } n \notin I \quad \text{or } 1 \in I, \\ f_{I,1} + \alpha f_{I,n} & \text{for } j = 1 \text{ when } n \notin I \quad \text{or } 1 \in I, \\ f_{I,j} + \alpha f_{\sigma(I),j} & \text{for } j \neq 1 \text{ when } n \in I \text{ and } 1 \notin I, \\ f_{I,1} + \alpha f_{I,n} + \alpha f_{\sigma(I),1} + \alpha^2 f_{\sigma(I),n} & \text{for } j = 1 \text{ when } n \in I \text{ and } 1 \notin I. \end{cases}$$

Suppose by way of contradiction that ℓ^2 does not divide $f_{I,n}$. Recall again that ℓ divides $f_{I,n}$ by part a) so some monomial M of degree 1 in x_n appears in $f_{I,n}$ with nonzero coefficient $c \in \mathbb{F}$. As $t_1(f_{I,n}) = f_{I,n}$ by Eq. (A.5),

$$f_{I,n} \in S^{G'} = \mathbb{F}[x_1^2 + x_1 x_n, x_i : i \neq 1] \quad \text{for } G' = \langle t_1 \rangle$$

(see [21]), and thus the degree of M in x_1 is even (as x_n divides $f_{I,n}$).

We analyze the coefficients of M and Mx_1/x_n in $f_{I,1}$, $f_{I,n}$, $f_{\sigma(I),1}$, and $f_{\sigma(I),n}$. Note that

$$f_{I,n} = 0 \cdot Mx_1/x_n + c \cdot M + \text{other terms}\,.$$

Next, $t_1(f_{I,1}) = f_{I,1} + f_{I,n}$ and $t_1(f_{\sigma(I),n}) = f_{\sigma(I),n} + f_{I,n}$ by Eq. (A.5), so the coefficients of Mx_1/x_n in $f_{I,1}$ and $f_{\sigma(I),n}$ are both equal to c by Lemma A.3. Fix $c', c'' \in \mathbb{F}$ with

$$f_{I,1} = c \cdot Mx_1/x_n + c' \cdot M + \text{other terms}, \quad f_{\sigma(I),n} = c \cdot Mx_1/x_n + c'' \cdot M + \text{other terms}\,.$$

Now we examine $f_{\sigma(I),1}$. On one hand, by Eq. (A.5) and Eq. (A.7),

$$t_1(f_{\sigma(I),1}) + f_{\sigma(I),1} = f_{\sigma(I),n} + f_{I,1} + f_{I,n} = 0 \cdot Mx_1/x_n + (c + c' + c'') \cdot M + \text{other terms and}$$
$$t_1^{(\alpha)}(f_{\sigma(I),1}) + f_{\sigma(I),1} = \alpha f_{\sigma(I),n} + \alpha f_{I,1} + \alpha^2 f_{I,n} = 0 \cdot Mx_1/x_n + (\alpha^2 c + \alpha c' + \alpha c'')M + \text{other terms}\,.$$

Let C be the coefficient of Mx_1/x_n in $f_{\sigma(I),1}$. Then, on the other hand, since Mx_1/x_n has odd degree in x_1,

$$t_1(f_{\sigma(I),1}) + f_{\sigma(I),1} = C \cdot M + \text{other terms}, \quad t_1^{(\alpha)}(f_{\sigma(I),1}) + f_{\sigma(I),1} = \alpha C \cdot M + \text{other terms},$$

so $C = c + c' + c''$ and $\alpha C = \alpha^2 c + \alpha c' + \alpha c''$, which implies that $c = 0$ (as $\alpha \neq 0, 1$), giving a contradiction. So ℓ^2 divides $f_{I,n}$. This completes part f) and the proof of the lemma. ☐

The next lemma is used to establish Lemma 3.7. We set $\delta = \delta_H$, which records when G comprises only one transvection and the identity (see Eq. (3.3)).

Lemma A.8. *For any set $\mathcal{B}$ of $n\binom{n}{k}$ elements in $(S \otimes \wedge^k V^* \otimes V)^G$, the determinant of* $\mathrm{Coef}(\mathcal{B})$ *is divisible by ℓ to the power*

$$\tbinom{n-1}{k} + (e-1)(n-1)\tbinom{n-1}{k-1} + e\Big((n-\delta)\big(\tbinom{n-1}{k} - \tbinom{n-b-1}{k}\big) + \tbinom{n-1}{k-1} - \tbinom{n-b-1}{k-1}\Big).$$

Proof. The claim follows immediately from Lemma A.4:

- ℓ divides each column in a set A of $\binom{n-1}{k}$ columns,
- ℓ^e divides each column in a set B of $\binom{n-1}{k-1} - \binom{n-b-1}{k-1}$ columns,
- ℓ^{e-1} divides each column in a set C of $(n-1)\binom{n-1}{k-1}$ columns,
- ℓ^e divides each column in a set D of $(n-1)\binom{n-1}{k} - (n-1)\binom{n-b-1}{k} - b\binom{n-b-1}{k-1}$ columns,
- ℓ^e divides each column in a set E of $b\binom{n-b-1}{k-1}$ columns after some column operations, and
- $\ell^{e(1-\delta)+1}$ divides each column in a set F of $\binom{n-1}{k} - \binom{n-b-1}{k}$ columns,

where the sets A, B, C, D, E are pairwise distinct and $F \subset A$. Hence, $\det \mathrm{Coef}(\mathcal{B})$ is divisible by ℓ to the power $|A| + (e-1)|C| + e\big(|B| + |D| + |E| + (1-\delta)|F|\big)$. ☐

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